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ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD

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This report has been reviewed by the Office of Public Affairs (PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A five-spherical layer plus core sphere, approximating the human or animal head, is exposed to plane wave, nonionizing electromagnetic radiation. The resulting induced fields within the simulated cranial structure are used in calculating the internal absorbed-power density distributions, average absorbed-power density and total absorbed power. The mathematical theory and formulas basic to accomplishing the computations are discussed in depth. Calculation requirements encountered are implemented in the form of a computer program. Discussion ⟶

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20. ABSTRACT (Continued)

of this users-oriented program covers such details as: structure and sequence of control parameter and data cards, output formats, and subroutine and function subprograms. Sample printouts and plots (linear and contour) of computer results and a listing of the FORTRAN source program are included. ⟵

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ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD

INTRODUCTION

The head is modeled by several homogeneous regions of tissue bounded by one or two members of a family of concentric spheres. These tissues include brain, cerebrospinal fluid (CSF), dura, bone, subcutaneous fat, and skin tissue. We assume that this complex of biological material is exposed to nonionizing electromagnetic radiation taking the form of a time-harmonic plane wave of peak amplitude, E_0. The time variation factor, $\exp(-i\omega t)$, has been suppressed in most of the discussion. Wave propagation is in the positive z-direction, and the electric field, E, is linearly polarized in the x-direction (Fig. 1). A rectangular-spherical coordinate system with origin at the center of an inner core sphere is used. Also, the medium surrounding the concentric spherical model is taken as free space (or vacuum). Thus our embedding medium is a nonconductor, and both the surrounding medium and the model are nonmagnetic. Each region $(p = 1, \ldots, N-1)$ into which the model is partitioned is homogeneous, isotropic, and possesses a unique dielectric constant and conductivity. All magnetic permeabilities are considered to have the value unity. The value "p = N" is reserved for reference to the containing medium.

The need for a multilayer model and the inadequacy of (1) ignoring the relatively thin outer structures, or (2) carrying out a volume average of the electrical properties of the regions can be seen by looking at Figure 2 for the case of the rhesus monkey. There graphically displayed, for comparison purposes, are three superimposed distributions of absorbed-power density along the z-axis. The monkey-head models consist of (1) pure brain tissue, (2) tissue with average volume of electrical properties of the structural components in Figure 1, and (3) unique tissues represented in Figure 1. The predicted distributions are based on 1-volt-per-meter intensity incident wave at 3 GHz.

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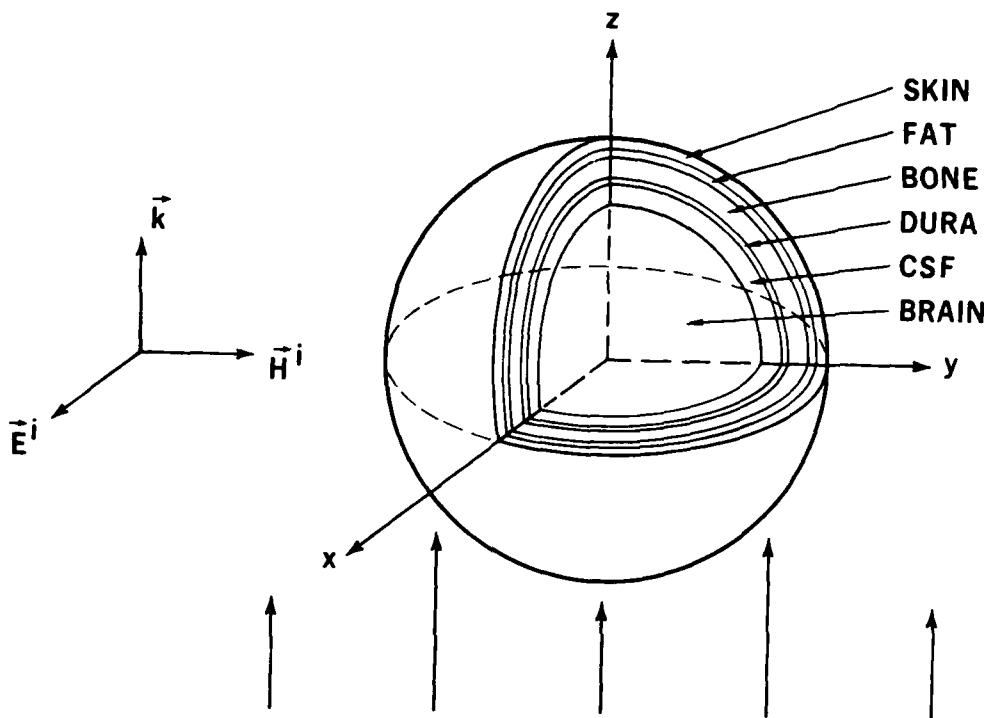
Figure 1. Electromagnetic plane wave impinging on a head
model composed of an inner core sphere and five
spherical shells.

Dimensions of structural media and electrical parameter values were
extracted from a table that was produced by Shapiro et al. (13). Table
1 presents such information. Contour plots--Figures 3 (ϕ = 0), 4 (ϕ =
$\pi/2$), and 5--are likewise based on information offered by Table 1.

The linear plot, Figure 6, and the contour plots--Figures 7 (ϕ = 0),
8 (ϕ = $\pi/2$), and 9--are founded on the entries of Table 2, an extraction
from a paper by Weil (15). Incident plane-wave characteristics are
1-volt-per-meter intensity and 1-GHz frequency. Other parameter values
pertinent to the computations for graphical construction are given in
Table 2.

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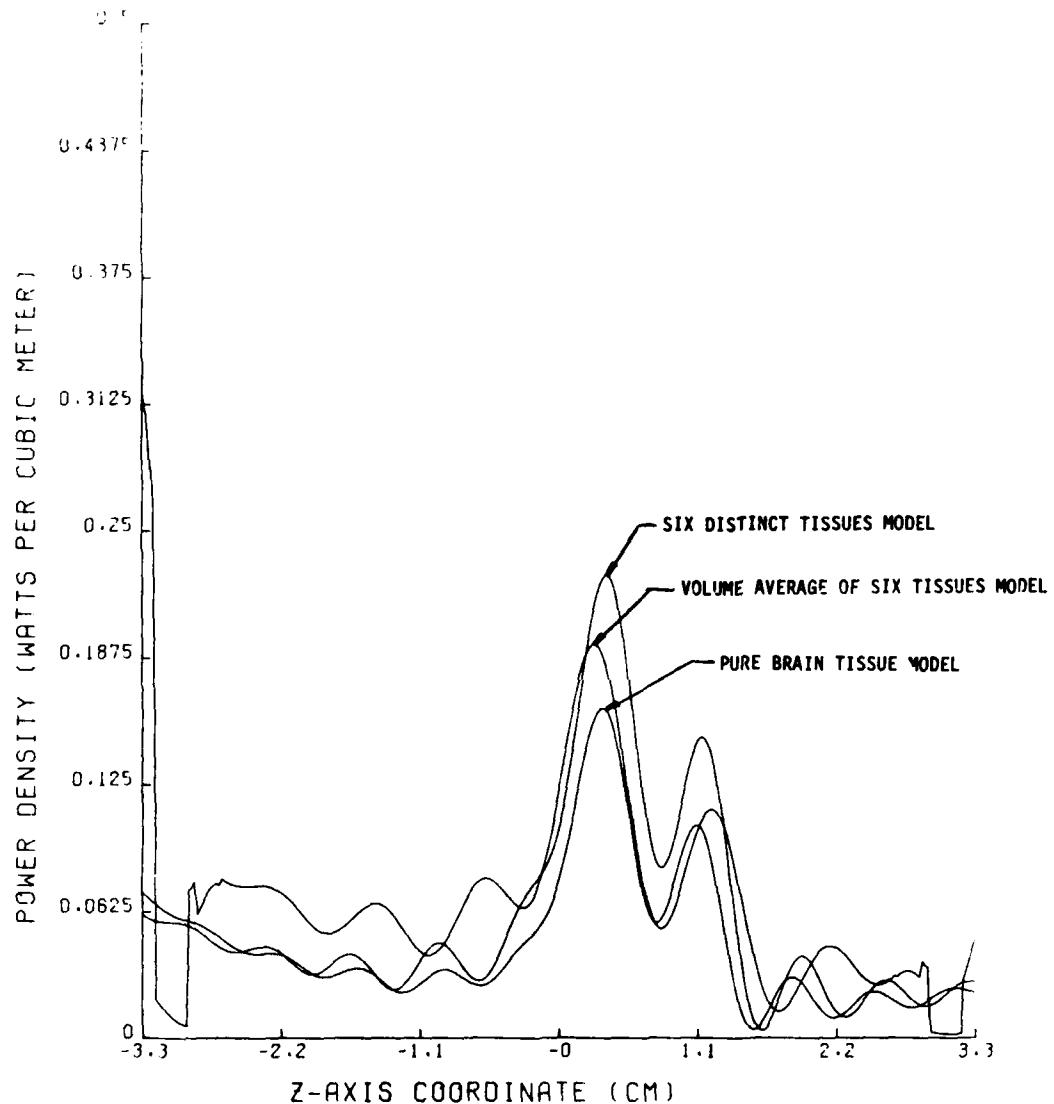
Figure 2. Distribution of power density along the z-axis for three different head models of the rhesus monkey. Spheres are of 3.3-cm radius and frequency is at 3 GHz.

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TABLE 1. RHESUS-MONKEY-HEAD DATA

Region (p)	Tissue modeled	Thickness (cm)	Radius of surface boundary, r_p (cm)	Relative dielectric constant,[a] ϵ_p	Conductivity,[a] σ_p (mho/m)
1	Brain	sphere	2.68	42.0	2.0
2	CSF	0.20	2.88	77.0	1.9
3	Dura	0.05	2.93	45.0	2.5
4	Bone	0.20	3.13	5.0	0.2
5	Fat	0.07	3.20	5.0	0.2
6	Skin	0.10	3.30	45.0	2.5

[a]At $T = 37^\circ C$ and $f = 3$ GHz.

TABLE 2. IDEALIZED HUMAN-HEAD DATA

Region (p)	Tissue modeled	Thickness (cm)	Radius of surface boundary, r_p (cm)	Relative dielectric constant,[a] ϵ_p	Conductivity,[a] σ_p (mho/m)
1	Brain	sphere	9.10	60.00	0.90
2	CSF	0.20	9.30	76.00	1.70
3	Dura	0.05	9.35	45.00	1.00
4	Bone	0.40	9.75	8.50	0.11
5	Fat	0.15	9.90	5.50	0.08
6	Skin	0.10	10.00	45.00	1.00

[a]At $f = 1$ GHz.

POWER DENSITY
(W/M^3)

⋈ 0.295

Y 0.265

Z 0.235

X 0.205

⊕ 0.175

◇ 0.145

✕ 0.115

+ 0.084

△ 0.055

⊘ 0.025
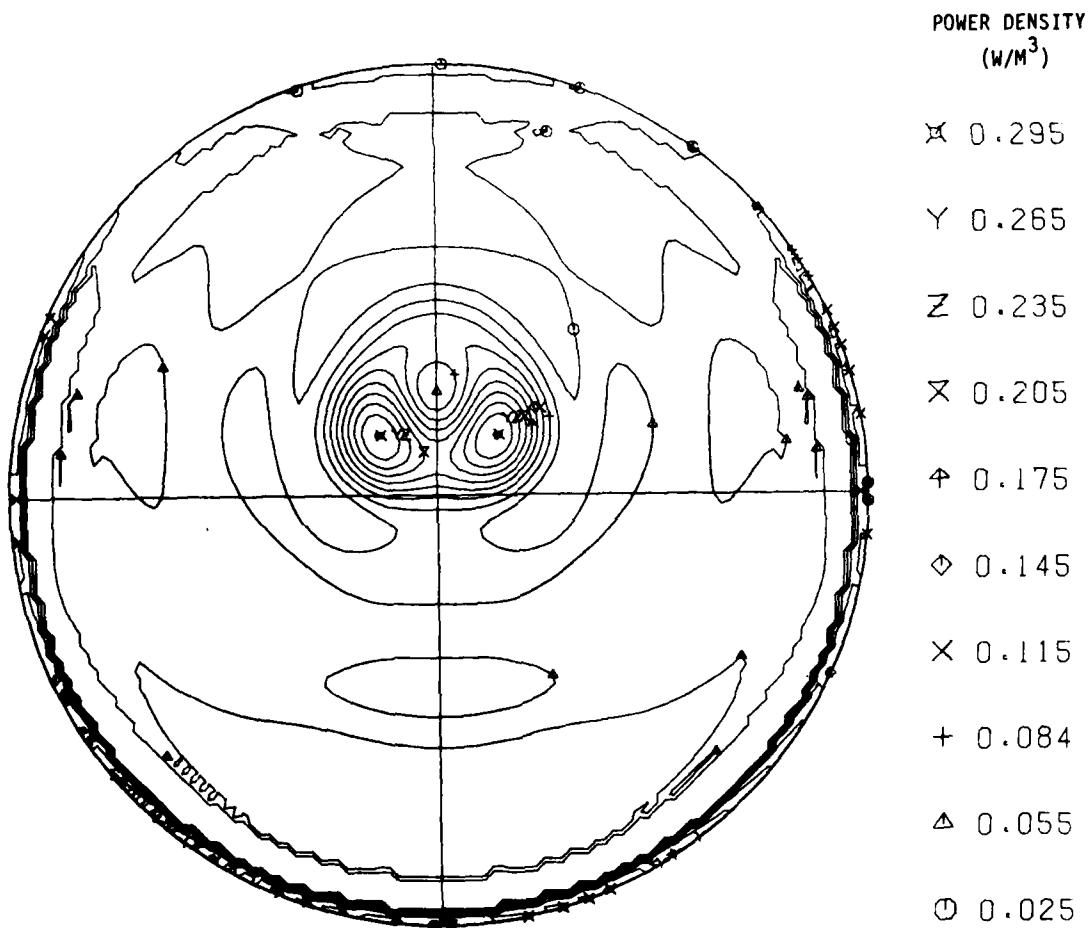
Figure 3.   Distribution of power density in a 3.3-cm-radius
sphere, a model of the rhesus-monkey head, at
3 GHz. (E-plane)

POWER DENSITY
(W/M³)

⬚ 0.295

Y 0.265

Z 0.235

X 0.205

♠ 0.175

◇ 0.145

× 0.115

+ 0.084

△ 0.055

① 0.025
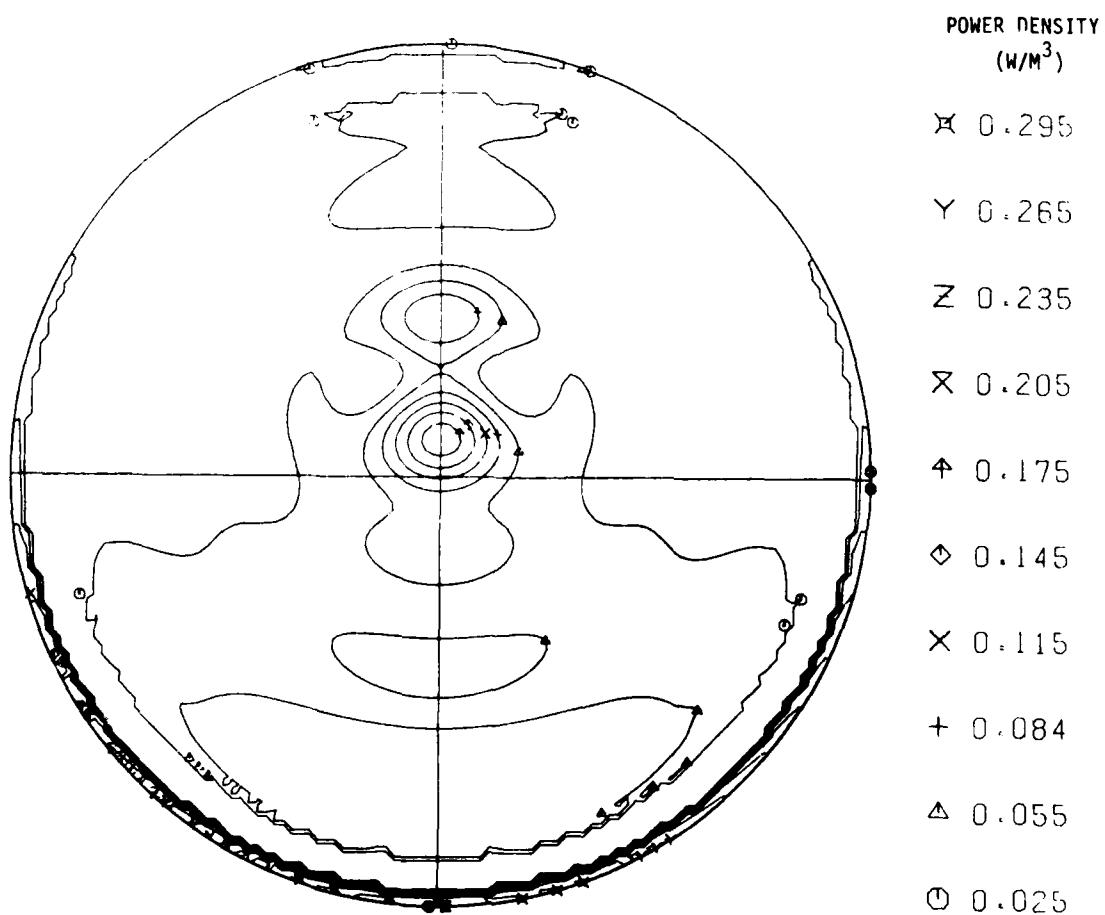
Figure 4. Distribution of power density in a
3.3-cm-radius sphere, a model of the
rhesus-monkey head, at 3 GHz. (H-plane)

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POWER DENSITY
(W/M³)

⌷ 0.1265

Y 0.1135

Z 0.1005

X̄ 0.0874

✢ 0.0744

◇ 0.0615

X 0.0485

+ 0.0355

△ 0.0225

⊙ 9.43999*10³
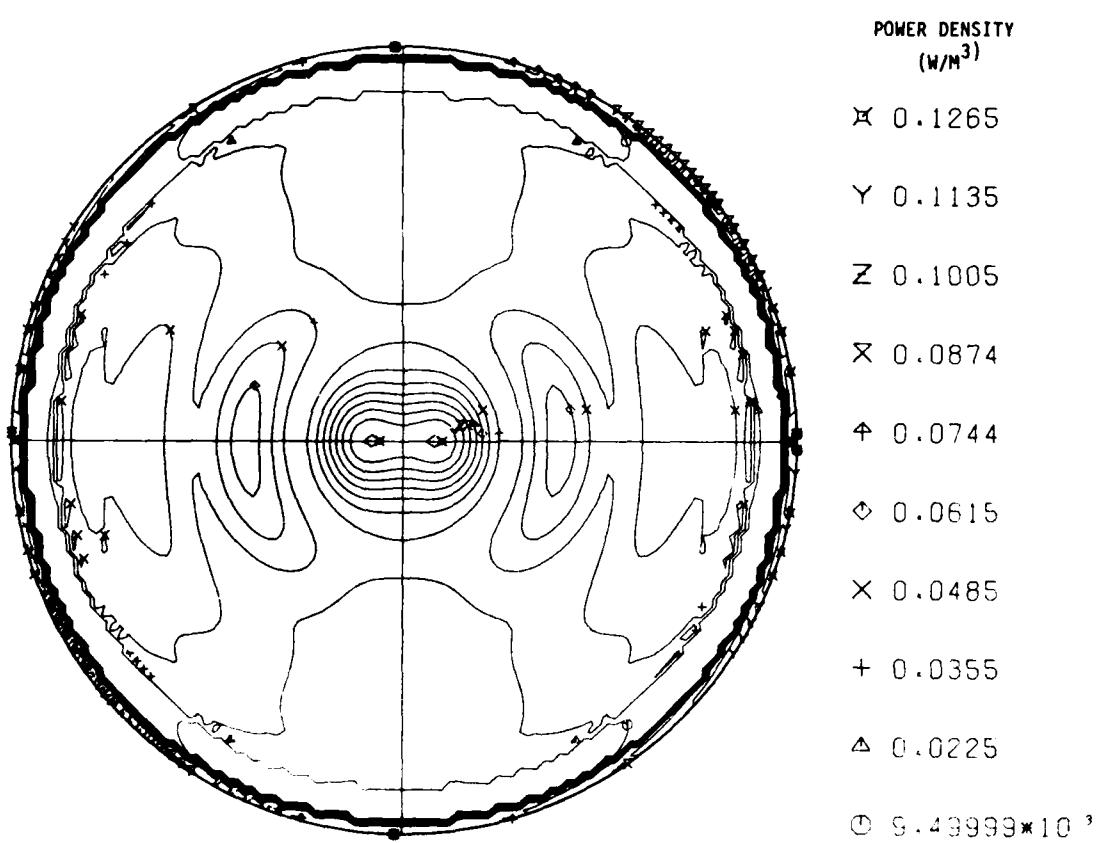
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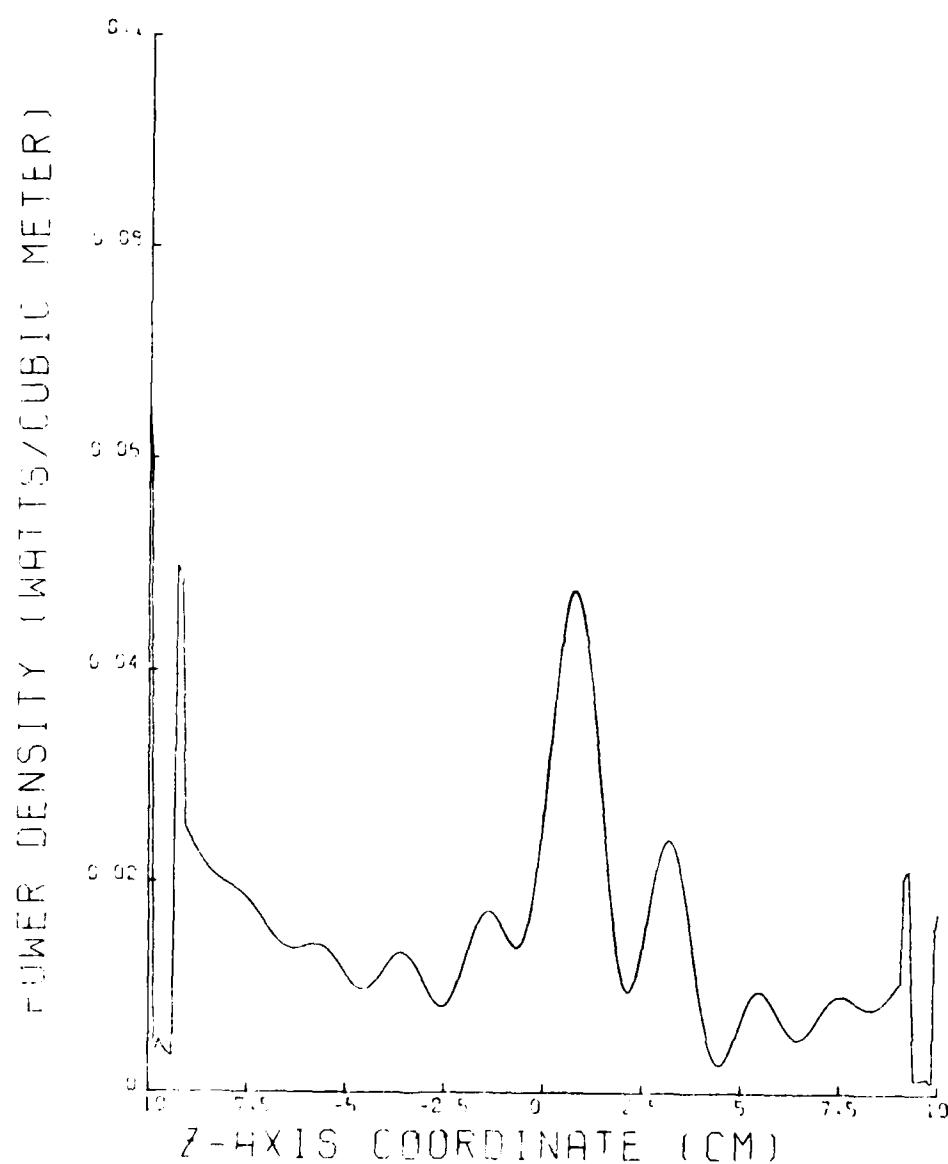
Figure 6.   Distribution of power density in a
10-cm-radius sphere, a model of the
human head, at 1 GHz along the z-axis.

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POWER DENSITY
(W/M³)

⋈ 0.065

Y 0.059

Z 0.052

✕ 0.045

✦ 0.038

◇ 0.031

✕ 0.024

+ 0.017

△ 0.01

⊕ 1.0*10⁻²
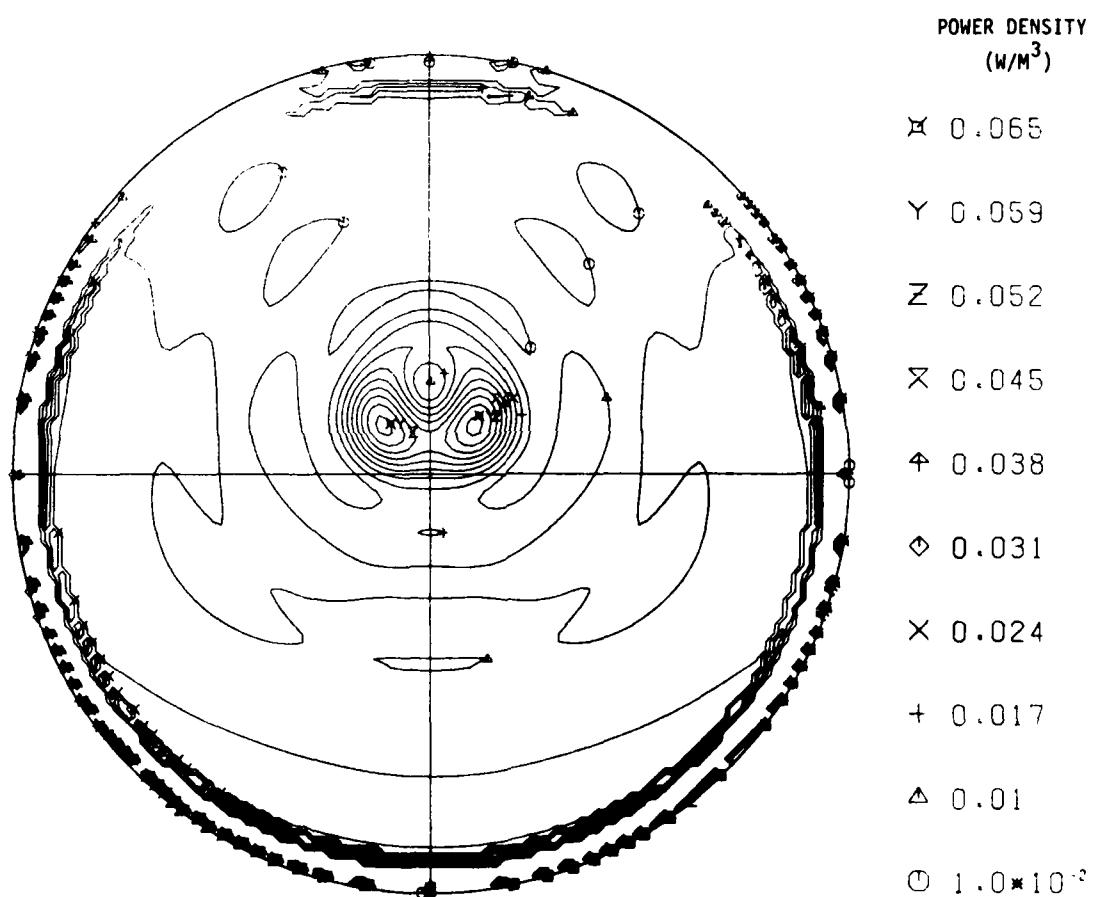
Figure 7. Distribution of power density in a
10-cm-radius sphere, a model of the
human head, at 1 GHz. (E-plane)

POWER DENSITY
(W/M³)

⋈ 0.0595

Y 0.0535

Z 0.0475

X 0.0415

✦ 0.0355

◇ 0.0295

✕ 0.0235

+ 0.0175

△ 0.0115

⊙ 5.5*10⁻³
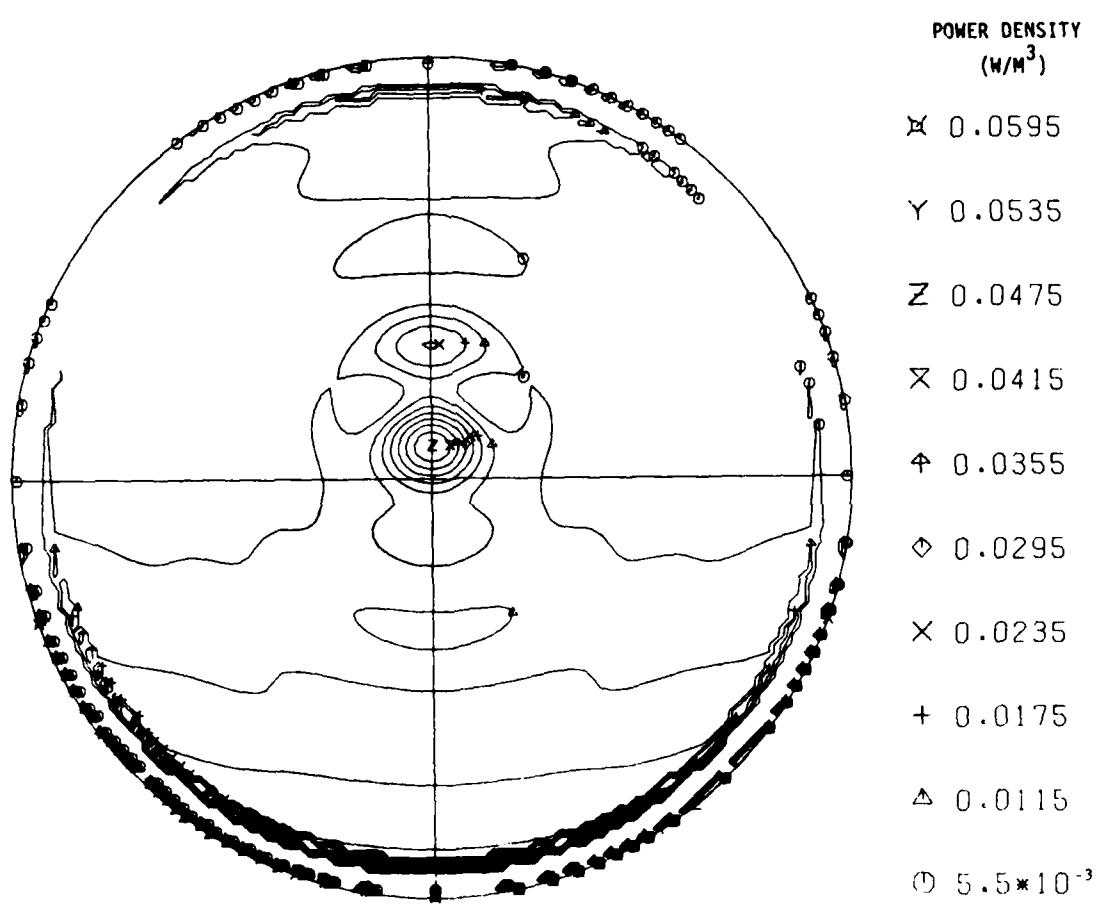
Figure 8. Distribution of power density in a
10-cm-radius sphere, a model of the
human head, at 1 GHz. (H-plane)

POWER DENSITY
(W/M³)

⋈	0.03215
Y	0.02885
Z	0.02555
⋈	0.02225
♠	0.01895
♦	0.01565
X	0.01235
+	$9.04999*10^{-3}$
△	$5.74999*10^{-3}$
①	$2.45*10^{-3}$
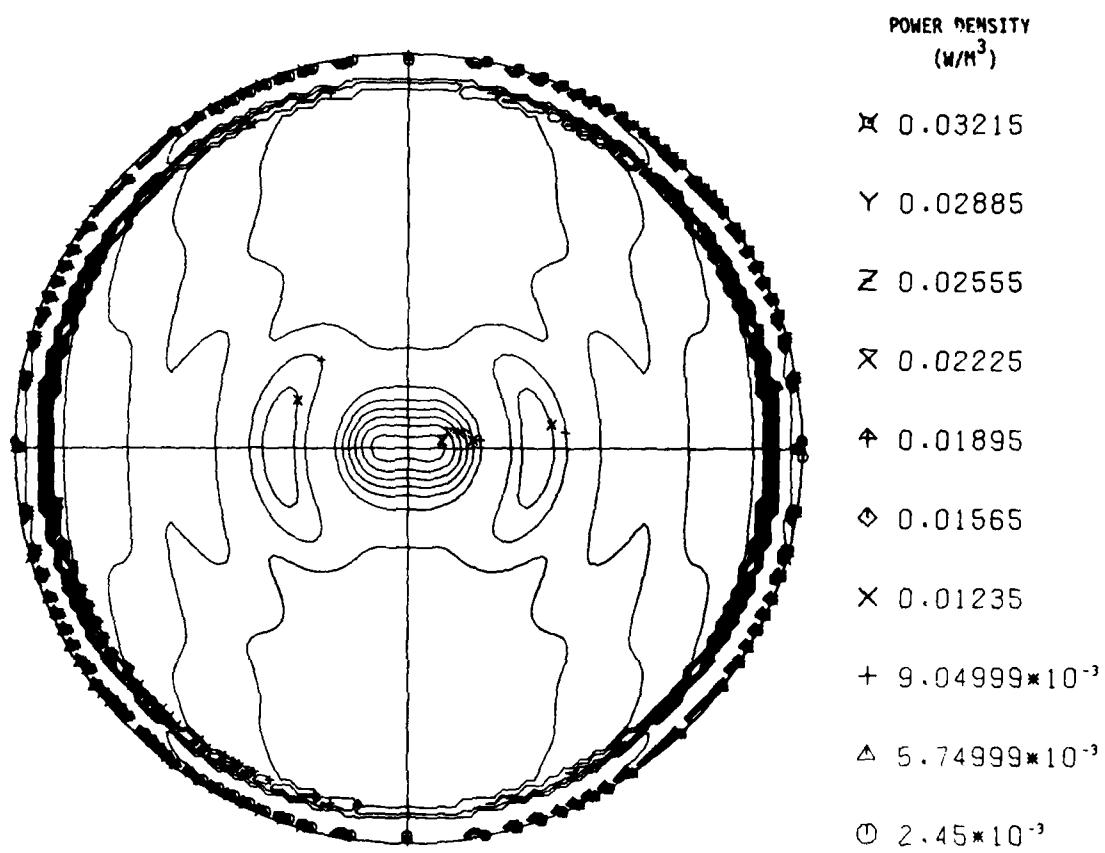
Figure 9. Distribution of power density in a
10-cm-radius sphere, a model of the
human head, at 1 GHz. (X,Y-plane)

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Nonuniform distribution of the absorbed energy inside of spheres gives rise to the internal appearance of "hot spots." Kritikos and Schwan (5) showed that "hot spots" appear in lossy, homogeneous, brain-like material spheres for the range of values of the radius and frequency: 0.1<R<8 cm and 300<f<1200 MHz. Shapiro et al. (13) and Weil (15) have shown the existence of hot spots inside multilayered spheres. Each has taken into account the importance of the frequency dependence of the sphere electrical properties. What precise conditions will precipitate the phenomenon are still not well known. We do know that the radius of the sphere and frequency, among others, do play a significant role. Occurrence takes place at the front surface or inside the head. It is a phenomenon that is importantly connected to small animals and infants.

The concentric spherical model represents one step forward in approximating reality as compared to the single, homogeneous sphere. Even so, the shortcomings of this model are to be found in (1) shape, (2) electrical properties, (3) thicknesses of tissue media, (4) assumption of tissue media being homogeneous and isotropic, and (5) inoperative conduction, convection, and radiation-heat-transfer mechanisms. The present computer program will be updated in the near future by an attempt to incorporate the mechanism of blood flow (along with other features). This may result in an appreciable reduction in the temperature rise now calculated.

The knowledge to be gleaned from this current research is directly related to the research effort of the Radiation Sciences Division at the USAF School of Aerospace Medicine. Briefly, here studies are being conducted to (1) determine the radiofrequency radiation-induced effects in biological specimens, (2) seek out possible hazards to personnel in a radiofrequency environment, (3) accurately measure and determine the distribution of energy in the whole biological body or just in a particular organ, (4) extrapolate response to radiation from the test animal

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to man in a meaningful manner, and (5) contribute in the design of realistic safety standards with a solid basis.

The division of this paper entitled "Mathematical Description" consists of four sections. Since spherical harmonics (Stratton [14], pp. 399-423) are used to expand the incident, induced, and scattered fields, we include in the section "Mathematical Preliminaries" details of the exact evaluation of inner products entering into the computation of expansion coefficients. In the section "Expansion of Induced Fields in Terms of Vector Wave Functions," the expansions are used to solve Maxwell's equations (Stratton [14], p. 26), subject to the condition that the tangential components of electric field $\vec{E}$ and magnetic field $\vec{H}$ are continuous across the spheres delimiting different regions of the head model. The section "Determination of Total Absorbed Power" considers the integrals that appear in Poynting's theorem. Such integrals are evaluated in closed form, thereby yielding a formula for the total power absorbed. The section "Summary of Key Equations and Formulas" contains a detailed summary of the formulas implemented by the computer program, Concentric Spherical Model (CSM), for automatically calculating the radiofrequency electromagnetic energy deposited in a concentric spherical model of the human or animal head.

The succeeding division, entitled "Program Description," contains pertinent information about the computer program that is beneficial to the user. The appendixes consist of a sample problem with computer results and a source listing of the program CSM.

To benefit users of this report, program CSM is described in sufficient depth to permit job setups and implementation on any modern computer. The mathematical theory and formulas basic in accomplishing the computations are discussed in an extensive manner. Discussion of this users-oriented computer program covers such details as structure and sequence of control parameter and data cards, output formats, and subroutine and function subprograms. Sample printouts and plots (linear and contour) of computer results and a listing of the FORTRAN IV source program are included.

MATHEMATICAL DESCRIPTION

Mathematical Preliminaries

This part of the mathematical discussion introduces an inner product on doubly periodic vector valued functions, presents a study of some of the properties of Legendre polynomials used in evaluating inner products, introduces vector wave functions, and verifies some of their properties.

Definition 1. *Let* $S = \{(\theta,\phi): 0 \leq \theta < \pi \text{ and } 0 \leq \phi < \pi\}$. *Then let* H(S,C) *denote the continuous complex valued functions* A *defined on* S *that satisfy the inequality*

$$||A||^2 = \int_0^{2\pi} [\int_0^{\pi} |A(\theta,\phi)|^2 \sin\theta d\theta] d\phi < \infty .$$ (1)

For any functions, A *and* B *in* H(S,C) *define the inner product* <,> *by the rule*

$$< A,B > = \int_0^{2\pi} [\int_0^{\pi} A(\theta,\phi)\overline{B(\theta,\phi)} \sin\theta d\theta] d\phi .$$ (2)

Proposition 1. *The space* H(S,C) *with the inner product* <,> *is a pre-Hilbert space.*

This follows immediately from the definition.

Now we need some properties of the associated Legendre functions.

Definition 2. *For all nonnegative integers,* m *and* n *define*

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!} D^{n+m}(x^2-1)^n .$$ (3)

Proposition 2. If m+n is even (odd), then $D^{n+m}(x^2-1)^n$ *is a linear combination of even (odd) powers of* x.

Proof. Observe that

$$(x^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{2k} (-1)^{n-k} . \tag{4}$$

Since an even (odd) number of derivatives of an even power of x is an even (odd) power of x, the proposition is true.

Corollary 1. If n+m is even (odd), then $p_n^m(x)$ *is an even (odd) function of* x.

Proof of Corollary 1. Since $(1-x^2)^{m/2}$ is an even function of x, the corollary follows immediately from Proposition 2.

Proposition 3. For all nonnegative integers m and n

$$I = \int_0^{\pi} P_n^m(\cos\theta)\left(\frac{d}{d\theta}\right)(P_n^m(\cos\theta))\sin\theta\,d\theta = 0 . \tag{5}$$

Proof. Since

$$\tfrac{1}{2}\left(\frac{d}{d\theta}\right)(P_n^m(\cos\theta)^2) = P_n^m(\cos\theta)\left(\frac{d}{d\theta}\right)(P_n^m(\cos\theta)) , \tag{6}$$

an integration by parts implies that

$$I = \tfrac{1}{2}[P_n^m(\cos\theta)^2\sin\theta\big|_0^{\pi} - \int_0^{\pi} P_n^m(\cos\theta)^2\cos\theta\,d\theta] . \tag{7}$$

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Substituting $x = \cos\theta$ and using the fact that

$$d\theta = -(1/\sqrt{1-x^2})dx,\tag{8}$$

it follows that

$$I = -\tfrac{1}{2}\int_{-1}^{1}P_n^m(x)^2(x/\sqrt{1-x^2})dx,\tag{9}$$

which in view of Corollary 1, implies that $I = 0$.

Proposition 4. For all nonnegative integers m *and* n, *we have that*

$$\int_0^\pi P_n^m(\cos\theta)P_r^m(\cos\theta)\sin\theta d\theta = \begin{cases} 0, & r \neq n \\ \dfrac{2(n+m)!}{(2n+1)(n-m)!}, & r = n. \end{cases}\tag{10}$$

Proof. The proof is carried out completely in Whittaker and Watson (16, pp. 324-325).

Proposition 5. For all nonnegative integers m, n, *and* r, *we have*

$$\int_0^\pi \frac{d}{d\theta}(P_n^m(\cos\theta))\frac{d}{d\theta}(P_r^m(\cos\theta))\sin\theta d\theta = A_{(n,r)}^m$$

$$= \delta_{(n,r)}\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}n(n+1)$$

$$+ \int_0^\pi (\frac{-m^2}{\sin^2\theta})P_n^m(\cos\theta)P_r^m(\cos\theta)\sin\theta d\theta.\tag{11}$$

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Proof. Observe that

$$\frac{d}{dx}\left[\frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n\right] = \frac{\frac{m}{2}(1-x^2)^{m/2-1}(-2x)}{2^n n!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$$

$$+ \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{n+m+1}}{dx^{n+m+1}}(x^2-1)^n. \qquad (12)$$

Let $(d/d\theta)F(\cos\theta) = F'(\cos\theta)(-\sin\theta)$. Thus, $x = \cos\theta$ implies that

$$\frac{dx}{d\theta}\frac{d}{dx} = \frac{d}{d\theta}. \qquad (13)$$

Hence

$$A^m_{(n,r)} = \int_0^\pi (-\sin\theta)P_n^{m\,'}(x)(-\sin\theta)P_r^{m\,'}(x)\sin\theta\, d\theta$$

$$= \int_1^{-1}(1-x^2)P_n^{m\,'}(x)P_r^{m\,'}(x)(-dx)$$

$$= \int_{-1}^{1}(1-x^2)P_n^{m\,'}(x)P_r^{m\,'}(x)dx. \qquad (14)$$

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Integrating by parts in the above integral, we find that

$$A^m_{(n,r)} = -\int_{-1}^{1} \frac{d}{dx}\left((1-x^2)\,\frac{d}{dx}\,P^m_n(x)\right)P^m_r(x)dx$$

$$= \int_{-1}^{1}\left[n(n+1) - \frac{m^2}{1-x^2}\right]P^m_n(x)P^m_r(x)dx$$

$$= \int_{-1}^{1}\left[\frac{-m^2}{1-x^2} + r(r+1)\right]P^m_n(x)P^m_r(x)dx. \tag{15}$$

Since $A^m_{(n,r)} = A^m_{(r,n)}$ the above relation shows that if $r \neq n$,

$$\int_{-1}^{1} P^m_n(x)P^m_r(x)dx = 0. \tag{16}$$

Substituting back $x = \cos\theta$, we find that

$$A^m_{(n,r)} = \delta_{(n,r)}\,\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,n(n+1)$$

$$+ \int_{0}^{\pi}\left(\frac{-m^2}{\sin^2\theta}\right)P^m_n(\cos\theta)P^m_r(\cos\theta)\sin\theta\,d\theta\ . \tag{17}$$

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Proposition 6. *For all nonnegative integers* m, n, *and* r, *we have*

$$\int_0^\pi [\frac{d}{d\theta}(P_n^m(\cos\theta)) \frac{d}{d\theta}(P_r^m(\cos\theta)) + (m^2/\sin^2\theta)P_n^m(\cos\theta)P_r^m(\cos\theta)]\sin\theta d\theta$$

$$= \delta_{(n,r)}(\frac{2}{2n+1}) \frac{(n+m)!}{(n-m)!} n(n+1) . \tag{18}$$

Proof of Proposition 6. From Proposition 5 we deduce equation 18.

Definition 3. *Let* $\vec{i}, \vec{j},$ *and* $\vec{k}$ *denote the unit vectors in the Cartesian coordinate system. Define*

$$\vec{e}_r = \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \tag{19}$$

$$\vec{e}_\theta = \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \tag{20}$$

and

$$\vec{e}_\phi = -\sin\phi\vec{i} + \cos\phi\vec{j} . \tag{21}$$

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Definition 4. *If* S *is a surface in* R^3 *bounded by a simple closed-curve* C, *and* $\vec{A}$ *is a* C^1 *vector field defined in a neighborhood of* S, *then* curl $(\vec{A})$ *is a vector field such that*

$$\int\int_S \text{curl}(\vec{A})\cdot\vec{N}d\sigma = \oint_C \vec{A}\cdot\vec{T}ds \ , \tag{22}$$

where $\vec{N}$ *and* $\vec{T}$ *are, respectively, the unit normals and the unit tangents of* S *and* C.

Proposition 7. *If* $\vec{A}$ *is a vector valued function of* r, θ, *and* φ, *then*

$$\text{curl}(\vec{A}) = \frac{1}{r\sin\theta}[(\partial/\partial\theta)(\sin\theta A_\phi) - (\partial/\partial\phi)A_\theta]\vec{e}_r$$

$$+ \frac{1}{r}[(1/\sin\theta)(\partial/\partial\phi)A_r - (\partial/\partial r)(rA_\phi)]\vec{e}_\theta$$

$$+ \frac{1}{r}[(\partial/\partial r)(rA_\theta) - (\partial/\partial\theta)A_r]\vec{e}_\phi \ . \tag{23}$$

Proof. This follows from Stokes' theorem and the fact that in Cartesian coordinates x,y,z, where $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$, the curl of vector $\vec{F}$ is defined by

$$\text{curl}(\vec{F}) = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\vec{i} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\vec{j}$$

$$+ (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\vec{k}. \tag{24}$$

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Proposition 8. *In spherical coordinates if* ψ *is a function of* r, θ, *and* ϕ, *then*

$$\Delta\psi = (1/r^2)[(\partial/\partial r)(r^2(\partial/\partial r)\psi)]$$

$$+ (1/(r^2\sin\theta))[(\partial/\partial\theta)(\sin\theta(\partial/\partial\theta)\psi)]$$

$$+ (1/(r^2\sin^2\theta))(\partial^2/\partial\phi^2)\psi . \tag{25}$$

Proof. This follows from the fact that in Cartesian coordinates

$$\Delta\psi = (\partial^2/\partial x^2)\psi + (\partial^2/\partial y^2)\psi + (\partial^2/\partial z^2)\psi \tag{26}$$

and the coordinate transforms

$$x = r\sin(\theta)\cos(\phi),$$

$$y = r\sin(\theta)\sin(\phi), \tag{27}$$

and

$$z = r\cos(\theta).$$

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Proposition 9. For any C^1 function ψ of r, θ, and ϕ,

$$\vec{A} = \psi r \vec{e}_r \tag{28}$$

implies that

$$\vec{M}_\psi = \text{curl}(\vec{A})$$

$$= (1/\sin\theta)((\partial/\partial\phi)\psi)\vec{e}_\theta - ((\partial/\partial\theta)\psi)\vec{e}_\phi . \tag{29}$$

Proof. This follows by direct substitution of equation 28 into equation 23.

Proposition 10. *Suppose* ψ *is a* C^2 *function satisfying*

$$\Delta\psi + k^2\psi = 0, \tag{30}$$

where k *is a complex number, then*

$$\vec{M}_\psi = \text{curl}(\psi r \vec{e}_r) \tag{31}$$

and

$$\vec{N}_\psi = (1/k)\text{curl}(\vec{M}_\psi) \tag{32}$$

imply that

$$\vec{N}_\psi = [(1/(kr))(\partial/\partial r)(r^2(\partial/\partial r)\psi) + kr\psi]\vec{e}_r$$

$$+ (1/(kr))(\partial^2/\partial\partial\theta)(r\psi)\vec{e}_\theta$$

$$+ (1/(kr\sin\theta))(\partial^2/\partial r\partial\phi)(r\psi)\vec{e}_\phi \; . \tag{33}$$

In the next section we work out some consequences of this proposition when the function ψ is the product of a spherical Bessel function, a Legendre polynomial, and a sine or a cosine.

Expansion of Induced Fields in Terms of Vector Wave Functions

In determining the response to a plane wave of a union of regions of dielectric material delimited by spheres, we use the vector wave functions (of k_p = complex propagation constant for the p-th layer of the dielectric material):

$$\vec{M}^{(e,j)}_{(1,n)} = - \frac{1}{\sin\theta} z_n^j(k_p r)P_n^1(\cos\theta)\sin\phi \; \vec{e}_\theta$$

$$- z_n^j(k_p r)(d/d\theta)(P_n^1(\cos\theta))\cos\phi \; \vec{e}_\phi \; , \tag{34}$$

$$\vec{M}_{(1,n)}^{(o,j)} = \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta) \cos\phi \ \vec{e}_\theta$$

$$- z_n^j(k_p r)(d/d\theta)(P_n^1(\cos\theta))\sin\phi \ \vec{e}_\phi \ , \tag{35}$$

$$\vec{N}_{(1,n)}^{(e,j)} = \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta)\cos\phi \ \vec{e}_r$$

$$+ \frac{1}{k_p r}(\partial/\partial r)(r z_n^j(k_p r))(d/d\theta)(P_n^1(\cos\theta))\cos\phi \ \vec{e}_\theta$$

$$- \frac{1}{k_p r \sin\theta}(\partial/\partial r)(r z_n^j(k_p r)) P_n^1(\cos\theta)\sin\phi \ \vec{e}_\phi \ , \tag{36}$$

and

$$\vec{N}_{(1,n)}^{(o,j)} = \frac{n(n+1)}{k_p r} z_n^j(k_p r) P_n^1(\cos\theta)\sin\phi \ \vec{e}_r$$

$$+ \frac{1}{k_p r}(\partial/\partial r)(r z_n^j(k_p r))(d/d\theta)(P_n^1(\cos\theta))\sin\phi \ \vec{e}_\theta$$

$$+ \frac{1}{k_p r \sin\theta}(\partial/\partial r)(r z_n^j(k_p r)) P_n^1(\cos\theta)\cos\phi \ \vec{e}_\phi \ . \tag{37}$$

Here

$$z_n^1(\rho) = j_n(\rho) = \sqrt{\pi/2\rho} \ J_{n+\frac{1}{2}}(\rho), \tag{38}$$

$$z_n^3(\rho) = h_n^1(\rho) = \sqrt{\pi/2\rho} \ H_{n+\frac{1}{2}}^1(\rho), \tag{39}$$

$$H_{n+\frac{1}{2}}^1(\rho) = J_{n+\frac{1}{2}}(\rho) + i Y_{n+\frac{1}{2}}(\rho), \tag{40}$$

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and $J_{n+\frac{1}{2}}(\rho)$ and $Y_{n+\frac{1}{2}}(\rho)$ are the Bessel and Neuman functions of order half-an-odd integer, respectively.

Proposion 11. *For all nonnegative integers* m *and* n *and all integers* j *and* j´ *in* {1,2,3}, *we have*

$$< \vec{M}^{(e,j)}_{(1,n)} , \vec{M}^{(o,j´)}_{(1,m)} > = 0, \tag{41}$$

$$< \vec{M}^{(e,j)}_{(1,n)} , \vec{M}^{(e,j´)}_{(1,m)} > = A\delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!}$$

$$= < \vec{M}^{(o,j)}_{(1,n)} , \vec{M}^{(o,j´)}_{(1,n)} > , \tag{42}$$

where

$$A = \pi \, r^2 \, z_n^j(kr) z_n^{j´}(kr) , \tag{43}$$

$$< \vec{N}^{(o,j)}_{(1,n)} , \vec{N}^{(e,j´)}_{(1,m)} > = 0 , \tag{44}$$

$$< \vec{N}^{(o,j)}_{(1,n)} , \vec{N}^{(o,j´)}_{(1,m)} > = B\delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} \, n(n+1)+C_n$$

$$= < \vec{N}^{(e,j)}_{(1,n)} , \vec{N}^{(e,j´)}_{(1,m)} > , \tag{45}$$

$$B = \pi \left(\frac{1}{k_p r}\right)^2 (\partial/\partial r)(rz_n^j(k_p r))(\partial/\partial r)(rz_n^{j'}(k_p r)) \tag{46}$$

and

$$C_n = \frac{2\pi}{2n+1} \frac{(n+1)!}{(n-1)!} \frac{n^2(n+1)^2}{k_p^2 r^2} z_n^j(k_p r) z_n^{j'}(k_p r) \,, \tag{47}$$

$$< \vec{M}^{(o,j)}_{(1,n)} \,, \vec{N}^{(o,j')}_{(1,m)} > = 0 \,, \tag{48}$$

$$< \vec{M}^{(e,j)}_{(1,n)} \,, \vec{N}^{(e,j')}_{(1,m)} > = 0 \,, \tag{49}$$

$$< \vec{M}^{(o,j)}_{(1,n)} \,, \vec{N}^{(e,j')}_{(1,m)} > = 0 \,. \tag{50}$$

Proof. This follows from the definitions and the facts that

$$\int_0^\pi [(DP_n^1(\cos\theta))(DP_m^1(\cos\theta)) + \frac{1}{\sin^2\theta} P_n^1(\cos\theta)P_m^1(\cos\theta)]\sin\theta d\theta$$

$$= \delta_{(n,m)} \frac{2}{2n+1} \frac{(n+1)!}{(n-1)!} n(n+1) \tag{51}$$

and

$$\int_0^\pi [P_m^1(\cos\theta)D(P_n^1(\cos\theta)) + P_n^1(\cos\theta)D(P_m^1(\cos\theta))]d\theta = 0 \,, \tag{52}$$

where $D = d/d\theta$.

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Now we want to develop formulas relating the fields. Let us write the fields for the p-th region as

$$\vec{E}_p = E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,1)} - ib_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,1)}$$

$$+ \alpha_{(\ell,p)} \vec{M}_{(1,\ell)}^{(o,3)} - i\beta_{(\ell,p)} \vec{N}_{(1,\ell)}^{(e,3)}] \tag{53}$$

and

$$\vec{H}_p = - \frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,1)} + ia_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,1)}$$

$$+ \beta_{(\ell,p)} \vec{M}_{(1,\ell)}^{(e,3)} + i\alpha_{(\ell,p)} \vec{N}_{(1,\ell)}^{(o,3)}] \tag{54}$$

in terms of the spherical vector functions $\vec{M}_{(1,\ell)}^{(i,j)}$ and $\vec{N}_{(1,\ell)}^{(i,j)}$ [cf. Stratton (14, p. 564) for function definitions] and the complex propagation constant k_p. The tangential components of the fields are

$$(\vec{E}_p)_\theta = E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)} \frac{P_\ell^1(\cos\theta)}{\sin\theta}(\cos\phi)j_\ell(k_p r_p)$$

$$- ib_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(rj_\ell(k_p r)))_{r=r_p}((d/d\theta)(P_\ell^1(\cos\theta))) \cos\phi$$

$$+ \alpha_{(\ell,p)} \frac{P_\ell^1(\cos\theta)}{\sin\theta}(\cos\phi)h_\ell^1(k_p r_p)$$

$$- i\beta_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(rh_\ell^1(k_p r)))_{r=r_p}((d/d\theta)P_\ell^1(\cos\theta))\cos\phi], \tag{55}$$

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$$(\vec{E}_p)_\phi = E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)}(-j_\ell(k_p r_p)))((d/d\theta)(P_\ell^1(\cos\theta)))\sin\phi$$

$$- ib_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(r_p j(k_p r)))_{r=r_p}(\frac{P_\ell^1(\cos\theta)}{\sin\theta})\sin\phi$$

$$+ \alpha_{(\ell,p)}(-h_\ell^1(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\sin\phi$$

$$- i\beta_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(rh_\ell^1(k_p r)))\frac{P_\ell^1(\cos\theta)}{\sin\theta}]_{r=r_p} , \qquad (56)$$

$$(\vec{H}_p)_\theta = - \frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)}(- \frac{P_\ell^1(\cos\theta)}{\sin\theta})(\sin\phi)j_\ell(k_p r_p)$$

$$+ ia_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(rj_\ell(k_p r)))_{r=r_p}(d/d\theta)(P_\ell^1(\cos\theta))\sin\phi$$

$$+ \beta_{(\ell,p)}(- \frac{P_\ell^1(\cos\theta)}{\sin\theta})(\sin\phi)h_\ell^1(k_p r_p)$$

$$+ i\alpha_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(rh_\ell^1(k_p r)))_{r=r_p}(d/d\theta)(P_\ell^1(\cos\theta))\sin\phi] \qquad (57)$$

$$(\vec{H}_p)_\phi = - \frac{k_p}{\mu_0 \omega} E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)}[b_{(\ell,p)}(-j_\ell(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\cos\phi$$

$$+ ia_{(\ell,p)}(1/k_p r_p)((\partial/\partial r)(rj_\ell(k_p r)))_{r=r_p}(\frac{P_\ell^1(\cos\theta)}{\sin\theta})\cos\phi$$

$$+ \beta_{(\ell,p)}(-h_\ell^1(k_p r_p))((d/d\theta)(P_\ell^1(\cos\theta)))\cos\phi$$

$$+ i\alpha_{(\ell,p)}(-1/k_p r_p)((\partial/\partial r)(rh_\ell^1(k_p r)))_{r=r_p}(\frac{P_\ell^1(\cos\theta)}{\sin\theta})\cos\phi] \qquad (58)$$

The boundary conditions implying continuity of the tangential component of the electric vector in the θ-direction may be described by the rule

$$\pi E_0 \sum_{\ell=1}^{\infty} [A^+_{(\ell,p)} \frac{P^1_\ell(\cos\theta)}{\sin\theta} - iB^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta))$$

$$+ A^+_{(\ell,p)} \frac{P^1_\ell(\cos\theta)}{\sin\theta} - iB^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta))]$$

$$= \pi E_0 \sum_{\ell=1}^{\infty} [A^-_{(\ell,p+1)} \frac{P^1_\ell(\cos\theta)}{\sin\theta} - iB^-_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta))$$

$$+ A^-_{(\ell,p+1)} \frac{P^1_\ell(\cos\theta)}{\sin\theta} - iB^-_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta))]. \tag{59}$$

Here

$$C(\ell) = i^\ell \frac{2\ell+1}{\ell(\ell+1)} , \tag{60}$$

$$A^+_{(\ell,p)} = C(\ell)j_\ell(k_p r_p)a_{(\ell,p)} , \tag{61}$$

$$A^-_{(\ell,p+1)} = C(\ell)j_\ell(k_{p+1}r_p)a_{(\ell,p+1)} , \tag{62}$$

$$A^+_{(\ell,p)} = C(\ell)h^1_\ell(k_p r_p)\alpha_{(\ell,p+1)} , \tag{63}$$

$$A^-_{(\ell,p+1)} = C(\ell)h^1_\ell(k_{p+1}r_p)\alpha_{(\ell,p+1)} , \tag{64}$$

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$$\mathcal{B}^+_{(\ell,p)} = C(\ell)(1/k_p r_p)(\partial/\partial r)(rh^1_\ell(k_p r))_{r=r_p} \beta(\ell,p) \ , \tag{65}$$

$$\mathcal{B}^-_{(\ell,p+1)} = C(\ell)(1/k_{p+1} r_p)(\partial/\partial r)(rh^1_\ell(k_{p+1} r))_{r=r_p} \beta(\ell,p+1) \ , \tag{66}$$

$$B^+_{(\ell,p)} = C(\ell)(1/k_p r_p)(\partial/\partial r)(rj_\ell(k_p r))_{r=r_p} b(\ell,p) \ , \tag{67}$$

$$B^-_{(\ell,p+1)} = C(\ell)(1/k_{p+1} r_p)(\partial/\partial r)(rj_\ell(k_{p+1} r))_{r=r_p} b(\ell,p+1). \tag{68}$$

Letting

$$S_{(\ell,p)} = A^-_{(\ell,p+1)} + A^-_{(\ell,p+1)} - A^+_{(\ell,p)} - A^+_{(\ell,p)} \tag{69}$$

and

$$T_{(\ell,p)} = B^-_{(\ell,p+1)} + B^-_{(\ell,p+1)} - B^+_{(\ell,p)} - B^+_{(\ell,p)} \ , \tag{70}$$

we deduce that

$$\sum_{\ell=1}^{\infty} [S_{(\ell,p)} \frac{P^1_\ell(\cos\theta)}{\sin\theta} - iT_{(\ell,p)}(\partial/\partial\theta)P^1_\ell(\cos\theta)] = 0. \tag{71}$$

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Observe that $(E_\phi)_p = (E_\phi)_{p+1}$ implies that

$$\pi E_0 \sum_{\ell=1}^{\infty} [(-\overset{+}{A}_{(\ell,p)})(d/d\theta)P_\ell^1(\cos\theta)-i(-\overset{+}{B}_{(\ell,p)})\frac{P_\ell^1(\cos\theta)}{\sin\theta}$$

$$+(-\overset{+}{A}_{(\ell,p)})(d/d\theta)P_\ell^1(\cos\theta)-i(-\overset{+}{B}_{(\ell,p)})\frac{P_\ell^1(\cos\theta)}{\sin\theta}]$$

$$= \pi E_0 \sum_{\ell=1}^{\infty} [(-\overset{-}{A}_{(\ell,p+1)})(d/d\theta)P_\ell^1(\cos\theta)-i(-\overset{-}{B}_{(\ell,p+1)})\frac{P_\ell^1(\cos\theta)}{\sin\theta}$$

$$+(-\overset{-}{A}_{(\ell,p+1)})(d/d\theta)P_\ell^1(\cos\theta)-i(-\overset{-}{B}_{(\ell,p+1)})\frac{P_\ell^1(\cos\theta)}{\sin\theta}] . \qquad (72)$$

Thus,

$$\sum_{\ell=1}^{\infty} \{S_{(\ell,p)}[(d/d\theta)P_\ell^1(\cos\theta)]-iT_{(\ell,p)}[\frac{P_\ell^1(\cos\theta)}{\sin\theta}]\} = 0 \qquad (73)$$

and we conclude that $S_{(\ell,p)} = 0$ and $T_{(\ell,p)} = 0$.

Upon introducing the constants

$$\overset{\sim}{\overset{+}{A}}_{(\ell,p)} = a_{(\ell,p)}(1/k_p r_p)(\partial/\partial r)(r j_\ell(k_p r))\underset{r=r_p}{} C(\ell), \qquad (74)$$

$$\overset{\sim}{\overset{+}{B}}_{(\ell,p)} = b_{(\ell,p)} j_\ell(k_p r_p) C(\ell), \qquad (75)$$

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$$\widetilde{A}^+_{(\ell,p)} = \alpha_{(\ell,p)}(1/(k_p r_p))(\partial/\partial r)(r h^1_\ell(k_p r))\Big|_{r=r_p} C(\ell), \tag{76}$$

$$\widetilde{B}^+_{(\ell,p)} = \beta_{(\ell,p)} h^1_\ell(k_p r_p) C(\ell), \tag{77}$$

$$\widetilde{A}^-_{(\ell,p+1)} = a_{(\ell,p+1)}(1/(k_{p+1} r_p))(\partial/\partial r)(r j_\ell(k_{p+1} r))\Big|_{r=r_p} C(\ell), \tag{78}$$

$$\widetilde{B}^-_{(\ell,p+1)} = b_{(\ell,p+1)} j_\ell(k_{p+1} r_p) C(\ell), \tag{79}$$

$$\widetilde{A}^-_{(\ell,p+1)} = \alpha_{(\ell,p+1)}(1/(k_{p+1} r_p))(\partial/\partial r)(r h^1_\ell(k_{p+1} r))\Big|_{r=r_p} C(\ell), \tag{80}$$

$$\widetilde{B}^-_{(\ell,p+1)} = \beta_{(\ell,p+1)} h^1_\ell(k_{p+1} r_p) C(\ell), \tag{81}$$

and setting $(H_p)_\theta = (H_{p+1})_\theta$, we arrive at the equality

$$-\left(\frac{k_p}{\mu_0 \omega}\right)\pi E_o \sum_{\ell=1}^{\infty}[\widetilde{B}^+_{(\ell,p)}\left(\frac{-P^1_\ell(\cos\theta)}{\sin\theta}\right) + i\widetilde{A}^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta))$$

$$+ \widetilde{B}^+_{(\ell,p)}\left(-\frac{P^1_\ell(\cos\theta)}{\sin\theta}\right) + i\widetilde{A}^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta))]$$

$$= -\frac{k_{p+1}}{\mu_0 \omega}\pi E_o \sum_{\ell=1}^{\infty}[\widetilde{B}^-_{(\ell,p+1)}\left(-\frac{P^1_\ell(\cos\theta)}{\sin\theta}\right) + i\widetilde{A}^-_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta))$$

$$+ \widetilde{B}^-_{(\ell,p+1)}\left(-\frac{P^1_\ell(\cos\theta)}{\sin\theta}\right) + i\widetilde{A}^-_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta))]. \tag{82}$$

Now set

$$\tilde{S}_{(\ell,p)} = \tilde{B}^-_{(\ell,p+1)}k_{p+1} + \tilde{B}^-_{(\ell,p+1)}k_{p+1} - \tilde{B}^+_{(\ell,p)}k_p - \tilde{B}^+_{(\ell,p)}k_p \qquad (83)$$

and

$$\tilde{T}_{(\ell,p)} = \tilde{A}^-_{(\ell,p+1)}k_{p+1} + \tilde{A}^-_{(\ell,p+1)}k_{p+1} - A^+_{(\ell,p)}k_p - A^+_{(\ell,p)}k_p \; . \qquad (84)$$

Equations 82-84 yield

$$\sum_{\ell=1}^{\infty} [\tilde{S}_{(\ell,p)}\left(- \frac{P^1_\ell(\cos\theta)}{\sin\theta}\right) + i\tilde{T}_{(\ell,p)}(d/d\theta)P^1_\ell(\cos\theta)] \; = \; 0 \; . \qquad (85)$$

The boundary condition

$$(H_p)_\phi = (H_{p+1})_\phi \qquad (86)$$

is now utilized.

We observe that

$$-\left(\frac{k_p}{\mu\omega}\right)\pi E_o \sum^{\infty}[-\tilde{B}^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta)) + i\tilde{A}^+_{(\ell,p)}\left(\frac{P^1_\ell(\cos\theta)}{\sin\theta}\right)$$

$$-\tilde{B}^+_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta)) + i\tilde{A}^+_{(\ell,p)}\left(\frac{P^1_\ell(\cos\theta)}{\sin\theta}\right)] = (H_p)_\phi,$$

and $(H_{p+1})_\phi =$

$$- \frac{k_{p+1}}{\mu\omega} \pi E_o \sum_{\ell=1}^{\infty} [-\tilde{B}^-_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta)) + i\tilde{A}^-_{(\ell,p+1)}(\frac{P^1_\ell(\cos\theta)}{\sin\theta})$$

$$- \tilde{B}_{(\ell,p+1)}((d/d\theta)P^1_\ell(\cos\theta)) + i\tilde{A}^-_{(\ell,p+1)}(\frac{P^1_\ell(\cos\theta)}{\sin\theta})] \ . \tag{87}$$

Thus,

$$\sum_{\ell=1}^{\infty} [- \tilde{B}^-_{(\ell,p+1)}k_{p+1} - \tilde{B}^-_{(\ell,p+1)}k_{p+1} + \tilde{B}^+_{(\ell,p)}k_p + \tilde{B}^+_{(\ell,p)}k_p](d/d\theta)P^1_\ell(\cos\theta)$$

$$+ i \sum_{\ell=1}^{\infty} [\tilde{A}^-_{(\ell,p+1)}k_{p+1} + \tilde{A}^-_{(\ell,p+1)}k_{p+1} - \tilde{A}^+_{(\ell,p)}k_p - \tilde{A}^+_{(\ell,p)}k_p](\frac{P^1_\ell(\cos\theta)}{\sin\theta})$$

$$= 0 \ . \tag{88}$$

This implies that

$$\sum_{\ell=1}^{\infty} [\tilde{S}_{(\ell,p)}((d/d\theta)P^1_\ell(\cos\theta)) - i\tilde{T}_{(\ell,p)}[\frac{P^1_\ell(\cos\theta)}{\sin\theta}]] = 0 \ . \tag{89}$$

Now equations 85 and 89 are used to determine the values of $\tilde{S}_{(\ell,p)}$ and $\tilde{T}_{(\ell,p)}$. Multiplying both sides of equation 85 by $P^1_{\ell'}(\cos\theta)\sin\theta$ and both sides of equation 89 by $[(d/d\theta)P^1_{\ell'}(\cos\theta)]\sin\theta$ and integrating from 0 to π, results in equation

$$\sum_{\ell=1}^{\infty} \tilde{S}_{(\ell,p)} \int_0^{\pi} \left[\frac{P_\ell^1(\cos\theta) P_{\ell'}^1(\cos\theta)}{\sin^2\theta} + ((d/d\theta)P_\ell^1(\cos\theta))((d/d\theta)P_{\ell'}^1(\cos\theta)) \right] \sin\theta\, d\theta$$

$$-i \sum_{\ell=1}^{\infty} \tilde{T}_{(\ell,p)} \int_0^{\pi} [P_\ell^1(\cos\theta)((d/d\theta)P_{\ell'}^1(\cos\theta))$$

$$+ ((d/d\theta)P_\ell^1(\cos\theta))P_{\ell'}^1(\cos\theta)]\, d\theta = 0 \qquad (90)$$

which implies, in view of equations 51 and 52, that $\tilde{S}_{(\ell,p)} = 0$. By symmetry $\tilde{T}_{(\ell,p)} = 0$. We conclude that

$$B_{(\ell,p+1)} h_\ell^1(k_{p+1} r_p) k_{p+1} + b_{(\ell,p+1)} j_\ell(k_{p+1} r_p) k_{p+1}$$

$$= B_{(\ell,p)} h_\ell^1(k_p r_p) k_p + b_{(\ell,p)} j_\ell(k_p r_p) k_p . \qquad (91)$$

Now the associated relation derived from equating the tangential components of the $\vec{E}$ vector is, from equations 71 and 73, the following:

$$\frac{1}{k_{p+1} r_p} [(\partial/\partial r)(r j_\ell(k_{p+1} r))]_{r=r_p} b_{(\ell,p+1)}$$

$$+ (\frac{1}{k_{p+1} r_p})[(\partial/\partial r)(r h_\ell^1(k_{p+1} r))]_{r=r_p} \beta_{(\ell,p+1)}$$

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$$= \frac{1}{k_p r_p}[(\partial/\partial r)(rj_\ell(k_p r))]_{r=r_p}{}^b(\ell,\rho)$$

$$+ (\frac{1}{k_p r_p})[(\partial/\partial r)(rh_\ell^1(k_p r))]_{r=r_p}{}^\beta(\ell,p) \quad . \tag{92}$$

Remark. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}, \tag{93}$$

then the inverse of A *is given by*

$$A^{-1} = \begin{pmatrix} a_{22}/\Delta & -a_{12}/\Delta \\ -a_{21}/\Delta & a_{11}/\Delta \end{pmatrix}, \tag{94}$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$.

We define for the sake of economy several terms, following Shapiro et al. (13). Let

$$\xi^+(\ell,p) = \frac{1}{k_p r_p}(\partial/\partial r)(rh_\ell^1(k_p r))_{r=r_p} = \frac{1}{k_p r_p}[h_\ell^1(k_p r_p) + k_p r_p h_\ell^{1'}(k_p r_p)]$$

$$= \frac{1}{k_p r_p}(\partial/\partial \rho)(\rho h_\ell^1(\rho))_{\rho=k_p r_p} \quad , \tag{95}$$

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$$\xi^-_{(\ell,p+1)} = \frac{1}{k_{p+1}r_p} (\partial/\partial\rho)(\rho h^1_\ell(\rho))_{\rho=k_{p+1}r_p} \quad , \tag{96}$$

$$\eta^+_{(\ell,p)} = \frac{1}{k_p r_p} (\partial/\partial\rho)(\rho j_\ell(\rho))_{\rho=k_p r_p}$$

$$= \frac{1}{k_p r_p} (\partial/\partial r)(r j_\ell(k_p r))_{r=r_p} \quad , \tag{97}$$

$$\eta^-_{(\ell,p+1)} = \frac{1}{k_{p+1}r_p} (\partial/\partial\rho)(\rho j_\ell(\rho))_{\rho=k_{p+1}r_p} \quad , \tag{98}$$

$$j^+_{(\ell,p)} = j_\ell(k_p r_p) \quad , \tag{99}$$

$$j^-_{(\ell,p+1)} = j_\ell(k_{p+1}r_p) \quad , \tag{100}$$

$$h^+_{(\ell,p)} = h^1_\ell(k_p r_p) \quad , \tag{101}$$

and

$$h^-_{(\ell,p)} = h^1_\ell(k_{p+1}r_p) \quad . \tag{102}$$

Now the relations between the coefficients in matrix form are

$$\begin{pmatrix} j^-_{(\ell,p+1)}k_{p+1} & h^-_{(\ell,p+1)}k_{p+1} \\ \\ \eta^-_{(\ell,p+1)} & \xi^-_{(\ell,p+1)} \end{pmatrix} = \underline{B}^{(\ell,p+1)} \tag{103}$$

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and

$$
\begin{pmatrix} j^+_{(\ell,p)}k_p & h^+_{(\ell,p)}k_p \\ \\ n^+_{(\ell,p)} & \xi^+_{(\ell,p)} \end{pmatrix} = B^{(\ell,p)}_+ \cdot \tag{104}
$$

Observe that

$$
B^{(\ell,p)}_+ \begin{pmatrix} b_{(\ell,p)} \\ \\ \beta_{(\ell,p)} \end{pmatrix} = B^{(\ell,p+1)}_- \begin{pmatrix} b_{(\ell,p+1)} \\ \\ \beta_{(\ell,p+1)} \end{pmatrix} \cdot \tag{105}
$$

Thus, letting

$$
R_{(\ell,p)} = (B^{(\ell,p)}_+)^{-1} B^{(\ell,p+1)}_- \ , \tag{106}
$$

we deduce that since

$$
(B^{(\ell,p)}_+)^{-1} = \begin{pmatrix} \xi^+_{(\ell,p)}/k_p\Delta_p & -h^+_{(\ell,p)}/\Delta_p \\ \\ -n^+_{(\ell,p)}/k_p\Delta_p & j^+_{(\ell,p)}/\Delta_p \end{pmatrix} \ , \tag{107}
$$

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where

$$\Delta_p = j^+_{(\ell,p)}\xi^+_{(\ell,p)} - h^+_{(\ell,p)}n^+_{(\ell,p)} , \tag{108}$$

that

$$R_{(\ell,p)} = \begin{pmatrix} \xi^+_{(\ell,p)}/k_p\Delta_p & -h^+_{(\ell,p)}/\Delta_p \\ -n^+_{(\ell,p)}/k_p\Delta_p & j^+_{(\ell,p)}/\Delta_p \end{pmatrix} B_{-}^{(\ell,p+1)}$$

$$= \begin{pmatrix} R^{(\ell,p)}_{(1,1)} & R^{(\ell,p)}_{(1,2)} \\ R^{(\ell,p)}_{(2,1)} & R^{(\ell,p)}_{(2,2)} \end{pmatrix} . \tag{109}$$

Here

$$R^{(\ell,p)}_{(1,1)} = [\xi^+_{(\ell,p)}j^-_{(\ell,p+1)}(\frac{k_{p+1}}{k_p}) - h^+_{(\ell,p)}n^-_{(\ell,p+1)}]/\Delta_p , \tag{110}$$

$$R^{(\ell,p)}_{(1,2)} = [h^-_{(\ell,p+1)}\xi^+_{(\ell,p)}(\frac{k_{p+1}}{k_p}) - h^+_{(\ell,p)}\xi^-_{(\ell,p+1)}]/\Delta_p , \tag{111}$$

$$R^{(\ell,p)}_{(2,1)} = [-n^+_{(\ell,p)}j^-_{(\ell,p+1)}(\frac{k_{p+1}}{k_p}) + n^-_{(\ell,p+1)}j^+_{(\ell,p)}]/\Delta_p , \tag{112}$$

$$R^{(\ell,p)}_{(2,2)} = [-h^-_{(\ell,p+1)}n^+_{(\ell,p)}(\frac{k_{p+1}}{k_p}) + \xi^-_{(\ell,p+1)}j^+_{(\ell,p)}]/\Delta_p . \tag{113}$$

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Now on to the development of the matrices relating the a-α coefficients in layer p to those in layer $p+1$. The first relation is derived from equation 69 and the definitions 60-68, and is expressed, using the notation in equations 95-102, as

$$h^-_{(\ell,p+1)}\alpha_{(\ell,p+1)} + j^-_{(\ell,p+1)}a_{(\ell,p)}$$

$$- h^+_{(\ell,p)}\alpha_{(\ell,p)} - j^+_{(\ell,p)}a_{(\ell,p)} = 0 . \tag{114}$$

The next relation, derived from equations 74-81 and equations 83-90, takes the form of

$$\alpha_{(\ell,p+1)} \frac{1}{k_{p+1}r_p}(\partial/\partial r)(rh^1_\ell(k_{p+1}r))_{r=r_p} C(\ell)k_{p+1}$$

$$+ a_{(\ell,p+1)} \frac{1}{k_{p+1}r_p}(\partial/\partial r)(rj_\ell(k_{p+1}r))_{r=r_p} C(\ell)k_{p+1}$$

$$- \alpha_{(\ell,p)} \frac{1}{k_p r_p} (\partial/\partial r)(rh^1_\ell(k_p r))_{r=r_p} C(\ell)k_p$$

$$- a_{(\ell,p)} \frac{1}{k_p r_p} (\partial/\partial r)(rj_\ell(k_p r))_{r=r_p} C(\ell)k_p = 0 \tag{115}$$

which, after using the notation expressed by equations 95-102, may be written as

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$$\alpha_{(\ell,p+1)} \xi^-_{(\ell,p+1)} k_{p+1} + a_{(\ell,p+1)} \eta^-_{(\ell,p+1)} k_{p+1}$$

$$- \alpha_{(\ell,p)} \xi^+_{(\ell,p)} k_p - a_{(\ell,p)} \eta^+_{(\ell,p)} k_p = 0 . \tag{116}$$

Let us define

$$A_+^{(\ell,p)} = \begin{pmatrix} j^+_{(\ell,p)} & h^+_{(\ell,p)} \\ \eta^+_{(\ell,p)} k_p & \xi^+_{(\ell,p)} k_p \end{pmatrix} \tag{117}$$

and

$$A_-^{(\ell,p+1)} = \begin{pmatrix} j^-_{(\ell,p+1)} & h^-_{(\ell,p+1)} \\ \eta^-_{(\ell,p+1)} k_{p+1} & \xi^-_{(\ell,p+1)} k_{p+1} \end{pmatrix} \tag{118}$$

Define

$$Q^{(\ell,p)} = \left(A_+^{(\ell,p)} \right)^{-1} \left(A_-^{(\ell,p+1)} \right) \tag{119}$$

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and

$$\Delta(\ell,p) = j^+(\ell,p)\xi^+(\ell,p) - h^+(\ell,p)n^+(\ell,p) . \tag{120}$$

Observe that since $(A_+^{(\ell,p)})^{-1}$ is given by

$$(A_+^{(\ell,p)})^{-1} = \begin{pmatrix} \xi^+(\ell,p)k_p/k_p\Delta_p & -h^+(\ell,p)/k_p\Delta_p \\ -n^+(\ell,p)k_p/k_p\Delta_p & j^+(\ell,p)/k_p\Delta_p \end{pmatrix}$$

$$= \begin{pmatrix} \xi^+(\ell,p)/\Delta(\ell,p) & -h^+(\ell,p)/k_p\Delta(\ell,p) \\ -n^+(\ell,p)/\Delta(\ell,p) & j^+(\ell,p)/k_p\Delta(\ell,p) \end{pmatrix} , \tag{121}$$

it follows that

$$\det((A_+^{(\ell,p)})^{-1}) = \frac{1}{k_p\Delta(\ell,p)}$$

$$= \frac{1}{\det(A_+^{(\ell,p)})} \tag{122}$$

and furthermore

$$(A_+^{(\ell,p)})^{-1}A_-^{(\ell,p+1)} = \begin{pmatrix} Q_{(1,1)}^{(\ell,p)} & Q_{(1,2)}^{(\ell,p)} \\ Q_{(2,1)}^{(\ell,p)} & Q_{(2,2)}^{(\ell,p)} \end{pmatrix} , \tag{123}$$

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where

$$Q_{(1,1)}^{(\ell,p)} = [\xi_{(\ell,p)}^+ j_{(\ell,p+1)}^- - (\frac{k_{p+1}}{k_p}) h_{(\ell,p)}^+ n_{(\ell,p+1)}^-]/\Delta_{(\ell,p)} \; , \qquad (124)$$

$$Q_{(1,2)}^{(\ell,p)} = [\xi_{(\ell,p)}^+ h_{(\ell,p+1)}^- - (\frac{k_{p+1}}{k_p}) h_{(\ell,p)}^+ \xi_{(\ell,p+1)}^-]/\Delta_{(\ell,p)} \; , \qquad (125)$$

$$Q_{(2,1)}^{(\ell,p)} = [(\frac{k_{p+1}}{k_p}) j_{(\ell,p)}^+ n_{(\ell,p+1)}^- - n_{(\ell,p)}^+ j_{(\ell,p+1)}^-]/\Delta_{(\ell,p)} \; , \qquad (126)$$

and

$$Q_{(2,2)}^{(\ell,p)} = [(\frac{k_{p+1}}{k_p}) j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - n_{(\ell,p)}^+ h_{(\ell,p+1)}^-]/\Delta_{(\ell,p)} \; . \qquad (127)$$

Now we wish to use the transition matrices $Q^{(\ell,p)}$ and $R^{(\ell,p)}$ to get relations between the internal and the external coefficients. First, note that $\alpha_{(\ell,1)} = \beta_{(\ell,1)} = 0$ and $a_{(\ell,N)} = b_{(\ell,N)} = 1$, where N is the number of regions into which space is subdivided by N-1 spheres. Second, observe that

$$\begin{bmatrix} a_{(\ell,1)} \\ 0 \end{bmatrix} = Q^{(\ell,1)} Q^{(\ell,2)} \dots Q^{(\ell,N-1)} \begin{bmatrix} 1 \\ \alpha_{(\ell,N)} \end{bmatrix} \qquad (128)$$

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and

$$\begin{bmatrix} b_{(\ell,1)} \\ 0 \end{bmatrix} = R^{(\ell,1)}R^{(\ell,2)} \cdots R^{(\ell,N-1)} \begin{bmatrix} 1 \\ \beta_{(\ell,N)} \end{bmatrix} \qquad (129)$$

or setting

$$Q = Q^{(\ell,1)}Q^{(\ell,2)} \cdots Q^{(\ell,N-1)} \qquad (130)$$

and

$$R = R^{(\ell,1)}R^{(\ell,2)} \cdots R^{(\ell,N-1)} \;, \qquad (131)$$

we have the following relations

$$\begin{pmatrix} a_{(\ell,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} Q_{(1,1)} & Q_{(1,2)} \\ Q_{(2,1)} & Q_{(2,2)} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_{(\ell,N)} \end{pmatrix} \qquad (132)$$

and

$$\begin{pmatrix} b_{(\ell,1)} \\ 0 \end{pmatrix} = \begin{pmatrix} R_{(1,1)} & R_{(1,2)} \\ R_{(2,1)} & R_{(2,2)} \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{(\ell,N)} \end{pmatrix} \;. \qquad (133)$$

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Thus, we see that

$$\alpha_{(\ell,n)} = -Q_{(2,1)}/Q_{(2,2)} \tag{134}$$

and

$$a_{(\ell,1)} = Q_{(1,1)} - Q_{(1,2)}Q_{(2,1)}/Q_{(2,2)} \ . \tag{135}$$

Furthermore, once $\alpha_{(\ell,p)}$ and $a_{(\ell,p)}$ are determined, we obtain $\alpha_{(\ell,p+1)}$ and $a_{(\ell,p+1)}$ by the relation

$$\begin{pmatrix} a_{(\ell,p)} \\ \\ \alpha_{(\ell,p)} \end{pmatrix} = \begin{pmatrix} Q_{(1,1)}^{(\ell,p)} & Q_{(1,2)}^{(\ell,p)} \\ \\ Q_{(2,1)}^{(\ell,p)} & Q_{(2,2)}^{(\ell,p)} \end{pmatrix} \begin{pmatrix} a_{(\ell,p+1)} \\ \\ \alpha_{(\ell,p+1)} \end{pmatrix} . \tag{136}$$

Also, we deduce from equation 129 that

$$\beta_{(\ell,N)} = R_{(2,1)}/R_{(2,2)} \tag{137}$$

and

$$b_{(\ell,1)} = R_{(1,1)} - R_{(1,2)}R_{(2,1)}/R_{(2,2)} \ . \tag{138}$$

As before, once $\beta_{(\ell,p)}$ and $b_{(\ell,p)}$ are determined, we obtain $\beta_{(\ell,p+1)}$ and $b_{(\ell,p+1)}$ by the relation

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$$\begin{pmatrix} b_{(\ell,p)} \\ \\ \beta_{(\ell,p)} \end{pmatrix} = \begin{pmatrix} R^{(\ell,p)}_{(1,1)} & R^{(\ell,p)}_{(1,2)} \\ \\ R^{(\ell,p)}_{(2,1)} & R^{(\ell,p)}_{(2,2)} \end{pmatrix} \begin{pmatrix} b_{(\ell,p+1)} \\ \\ \beta_{(\ell,p+1)} \end{pmatrix}. \tag{139}$$

By repeated application of matrix equations 136 and 139, we determine all expansion coefficients; and thus, using equations 53 and 54, completely determine electric field $\vec{E}$ and magnetic field $\vec{H}$.

Determination of Total Absorbed Power

The Poynting vector is generally interpreted as a vector having length equal to the power per unit area traveling across a surface normal to the vector and direction of the power flow. One can show by the Gauss divergence theorem that the Poynting vector is given by

$$\vec{S} = \frac{\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}}{2} . \tag{140}$$

Maxwell's equations then imply that

$$div(\vec{S}) + \frac{\vec{J} \cdot \vec{J}^*}{\sigma} + (\partial/\partial t)(\frac{\varepsilon \vec{E} \cdot \vec{E}^* + \mu \vec{H} \cdot \vec{H}^*}{2}) = 0 , \tag{141}$$

Poynting's theorem in differential form in the absence of impressed electromotive forces for linear material media and where ε, μ, σ, and $\vec{J}$ are the permittivity, permeability, conductivity, and electric current density, respectively, and the sign $*$ attached to a vector denotes its complex conjugate.

Let us write (in terms of the spherical coordinate base vectors $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$)

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$$\vec{E} = E_r \, \vec{e}_r + E_\theta \, \vec{e}_\theta + E_\phi \, \vec{e}_\phi \tag{142}$$

and

$$\vec{H} = H_r \vec{e}_r + H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi \ , \tag{143}$$

where

$$\vec{e}_r = \sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k} \ , \tag{144}$$

$$\vec{e}_\theta = \cos\theta\cos\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j} - \sin\theta\,\vec{k} \ , \tag{145}$$

and

$$\vec{e}_\phi = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j} \ . \tag{146}$$

Observe that

$$\vec{e}_r \times \vec{e}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= \vec{i}(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi) - \vec{j}(-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi)$$
$$+ \vec{k}(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi) = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}$$

$$= \vec{e}_\phi \ . \tag{147}$$

A similar calculation shows that

$$\vec{e}_\theta \times \vec{e}_\phi = \vec{e}_r \tag{148}$$

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and

$$\vec{e}_\phi \times \vec{e}_r = \vec{e}_\theta \quad . \tag{149}$$

Thus

$$\vec{E} \times \vec{H} = (E_\theta H_\phi - H_\theta E_\phi)\vec{e}_r + (E_\phi H_r - H_\phi E_r)\vec{e}_\theta + (E_r H_\theta - H_r E_\theta)\vec{e}_\theta \quad . \tag{150}$$

What we need to compute is

$$\vec{S} \cdot (-\vec{N}) = (\vec{E} \times \vec{H}) \cdot (-\vec{N}) \tag{151}$$

When $\vec{N} = \vec{e}_r$. Now the power going into the sphere is

$$\int_0^\pi \left[\int_0^{2\pi} - (E_\theta H_\phi - H_\theta E_\phi)\sin\theta\, d\theta \right] d\phi \quad . \tag{152}$$

At this point, we stop to refamiliarize ourselves with the structure of the vector wave functions listed below:

$$\vec{M}_{(1,n)}^{(e,j)} = - \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta)\sin\phi\,\vec{e}_\theta$$
$$- z_n^j(k_p r)(d/d\theta)(P_n^1(\cos\theta))\cos\phi\,\vec{e}_\phi \quad , \tag{153}$$

$$\vec{M}_{(1,n)}^{(o,j)} = \frac{1}{\sin\theta} z_n^j(k_p r) P_n^1(\cos\theta)\cos\phi\,\vec{e}_\theta$$
$$- z_n^j(k_p r)(d/d\theta)(P_n^1(\cos\theta))\sin\phi\,\vec{e}_\phi \quad , \tag{154}$$

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$$\vec{N}^{(o,j)}_{(1,n)} = \frac{n(n+1)}{k_p r} z^j_n(k_p r) P^1_n(\cos\theta)\sin\phi\vec{e}_r$$

$$+ \frac{1}{k_p r} (\partial/\partial r)(rz^j_n(k_p r))(d/d\theta)(P^1_n(\cos\theta))\sin\phi\vec{e}_\theta$$

$$+ \frac{1}{k_p r\sin\theta}(\partial/\partial r)(rz^j_n(k_p r))P^1_n(\cos\theta)\cos\phi\vec{e}_\phi \ , \tag{155}$$

and

$$\vec{N}^{(e,j)}_{(1,n)} = \frac{n(n+1)}{k_p r} z^j_n(k_p r)P^1_n(\cos\theta)\cos\phi\vec{e}_r$$

$$+ \frac{1}{k_p r} (\partial/\partial r)(rz^j_n(k_p r))(d/d\theta)(P^1_n(\cos\theta))\cos\phi\vec{e}_\theta$$

$$- \frac{1}{k_p r\sin\theta}(\partial/\partial r)(rz^j_n(k_p r))P^1_n(\cos\theta)\sin\phi\vec{e}_\phi \ ; \tag{156}$$

to contemplate the use of the fact that in the region (p=N) surrounding
the biological material the electromagnetic field is

$$\vec{E} = E_o \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} [\vec{M}^{(o,1)}_{(1,n)} - i\vec{N}^{(e,1)}_{(1,n)}$$

$$+ \alpha_{(m,N)}\vec{M}^{(o,3)}_{(1,n)} - i\beta_{(n,N)}\vec{N}^{(e,3)}_{(1,n)}] \tag{157}$$

$$\vec{H} = - \frac{k_N}{\mu_o \omega} E_o \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} [\vec{M}^{(e,1)}_{(1,n)} + i\vec{N}^{(o,1)}_{(1,n)}$$

$$+ \beta_{(n,N)}\vec{M}^{(e,3)}_{(1,n)} + i\alpha_{(n,N)}\vec{N}^{(o,3)}_{(1,n)}] , \tag{158}$$

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or more compactly

$$\vec{E} = \vec{E}^i + \vec{E}^r \qquad (159)$$

and

$$\vec{H} = \vec{H}^i + \vec{H}^r \qquad (160)$$

and in accordance with Stratton (14, p. 568), we write (where a factor of ½ has been deleted and now the complex conjugate is indicated by overbar —)

$$\bar{S}_r = E_\theta \bar{H}_\phi - E_\phi \bar{H}_\theta \qquad (161)$$

and take

$$W_a = -Re \int_0^\pi [\int_0^{2\pi} \bar{S}_r \ r^2 \ \sin\theta d\phi] d\theta. \qquad (162)$$

Observe that

$$\frac{1}{2}(\vec{E} \times \vec{\bar{H}} + \vec{\bar{E}} \times \vec{H}) = Re(\vec{E} \times \vec{\bar{H}})$$

$$= Re(\vec{E}^i \times \vec{\bar{H}}^i) + Re(\vec{E}^i \times \vec{\bar{H}}^r + \vec{E}^r \times \vec{\bar{H}}^i)$$

$$+ Re(\vec{E}^r \times \vec{\bar{H}}^r). \qquad (163)$$

A direct calculation shows that

$$\oint Re(\vec{E}^i \times \vec{\bar{H}}^i) \cdot (-\vec{N})dA = 0 . \qquad (164)$$

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Thus, to get the total energy absorbed by the sphere, we need only compute

$$W_a = W_t - W_s,\qquad(165)$$

where W_t represents the energy dissipated as heat plus the scattered energy, and W_s represents the scattered energy.

Now

$$
\begin{aligned}
W_s &= \mathrm{Re}\int_0^{2\pi}\int_0^{\pi}(E_\theta\overline{H}_\phi - E_\phi\overline{H}_\theta)r^2\sin\theta\,d\theta\,d\phi \\[2mm]
&= \mathrm{Re}\int_0^{2\pi}\int_0^{\pi}\Big\{E_0\sum_{\ell=1}^{\infty}i^\ell\frac{2\ell+1}{\ell(\ell+1)}[\alpha_{(\ell,N)}(M_{(1,\ell)}^{(o,3)})_\phi - i\beta_{(\ell,N)}(N_{(1,\ell)}^{(e,3)})_\phi]\Big\} \\[2mm]
&\quad\cdot\Big\{-\frac{k_N}{\mu_0\omega}\overline{E}_0\sum_{s=1}^{\infty}\overline{i}^{\,s}\frac{2s+1}{s(s+1)}[\overline{\beta}_{(s,N)}(\overline{M}_{(1,s)}^{(e,3)})_\theta - i\overline{\alpha}_{(s,N)}(\overline{N}_{(1,s)}^{(o,3)})_\theta]\Big\}r^2\sin\theta\,d\phi \\[2mm]
&\quad-\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}\Big\{E_0\sum_{\ell=1}^{\infty}i^\ell\frac{2\ell+1}{\ell(\ell+1)}[\alpha_{(\ell,N)}(M_{(1,\ell)}^{(o,3)})_\theta - i\beta_{(\ell,N)}(N_{(1,\ell)}^{(e,3)})_\theta]\Big\} \\[2mm]
&\quad\cdot\Big\{\frac{k_N}{\mu_0\omega}\overline{E}_0\sum_{s=1}^{\infty}\overline{i}^{\,s}\frac{2s+1}{s(s+1)}[\overline{\beta}_{(s,N)}(\overline{M}_{(1,s)}^{(e,3)})_\phi - i\overline{\alpha}_{(s,N)}(\overline{N}_{(1,s)}^{(o,3)})_\phi]\Big\}r^2\sin\theta\,d\phi \\[2mm]
&= \sum_{s=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{(2s+1)(2\ell+1)}{s(s+1)\ell(\ell+1)}[F_{(\ell,s)}^{(\alpha,\alpha)}\alpha_{(\ell,N)}\overline{\alpha}_{(s,N)} + F_{(\ell,s)}^{(\alpha,\beta)}\alpha_{(\ell,N)}\overline{\beta}_{(s,N)} \\[2mm]
&\quad + F_{(\ell,s)}^{(\beta,\alpha)}\beta_{(\ell,N)}\overline{\alpha}_{(s,N)} + F_{(\ell,s)}^{(\beta,\beta)}\beta_{(\ell,N)}\overline{\beta}_{(s,N)}]\ .\qquad(166)
\end{aligned}
$$

Here

$$
^{\alpha}(\ell,N)^{\bar{\alpha}}(s,N) F_{(\ell,s)}^{(\alpha,\alpha)} = \text{Re} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{k_N}{\mu_o \omega} E_o^2 \ i^{s+\ell+1}(-1)^{s+1} {}^{\alpha}(\ell,N)^{\bar{\alpha}}(s,N)
$$

$$
\cdot [(M_{(1,\ell)}^{(o,3)})_\phi \overline{(N_{(1,s)}^{(o,3)})_\theta} - (M_{(1,\ell)}^{(o,3)})_\theta \overline{(N_{(1,s)}^{(o,3)})_\phi}] r^2 \sin\theta d\theta d\phi,
$$

$$(167)$$

$$
^{\alpha}(\ell,N)^{\bar{\beta}}(s,N) F_{(\ell,s)}^{(\alpha,\beta)} = \text{Re} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{k_N}{\mu_o \omega} E_o^2 \ i^{s+\ell}(-1)^{s} {}^{\alpha}(\ell,N)^{\bar{\beta}}(s,N)
$$

$$
\cdot [(M_{(1,\ell)}^{(o,3)})_\phi \overline{(M_{(1,s)}^{(e,3)})_\theta} - (M_{(1,\ell)}^{(o,3)})_\theta \overline{(M_{(1,\ell)}^{(e,3)})_\phi}] r^2 \sin\theta d\theta d\phi,
$$

$$(168)$$

$$
^{\beta}(\ell,N)^{\bar{\alpha}}(s,N) F_{(\ell,s)}^{(\beta,\alpha)} = \text{Re} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{k_N}{\mu_o \omega} E_o^2 \ i^{s+\ell}(-1)^{s} {}^{\beta}(\ell,N)^{\bar{\alpha}}(s,N)
$$

$$
\cdot [-(N_{(1,\ell)}^{(e,3)})_\phi \overline{(N_{(1,s)}^{(o,3)})_\theta} + (N_{(1,\ell)}^{(e,3)})_\theta \overline{(N_{(1,s)}^{(o,3)})_\phi}] r^2 \sin\theta d\theta d\phi.
$$

$$(169)$$

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Observe that

$$\int_0^{2\pi} \int_0^{\pi} \frac{k_N}{\mu_o \omega} E_o^2 \, i^{s+\ell} (-1)^s \beta_{(\ell,N)} \bar{\alpha}_{(s,N)}$$

$$[(- \frac{1}{k_N r \sin\theta}(\partial/\partial r)(r h_\ell^1(k_N r)) P_\ell^1(\cos\theta)\sin\phi)$$

$$\cdot (\frac{1}{k_N r} \overline{(\partial/\partial r)(r h_s^1(k_N r))}(d/d\theta) P_s^1(\cos\theta)\sin\phi$$

$$+ (\frac{1}{k_N r} (\partial/\partial r)(r h_\ell^1(k_N r))(d/d\theta) P_\ell^1(\cos\theta)\cos\phi)$$

$$\cdot (\frac{1}{k_N r \sin\theta} \overline{(\partial/\partial r)(r h_s^1(k_N r))} P_s^1(\cos\theta)\cos\phi)] \sin\theta \, d\theta \, d\phi$$

$$= \beta_{(\ell,N)} \bar{\alpha}_{(s,N)} F_{(\ell,s)}^{(\beta,\alpha)} \, . \tag{170}$$

Since for all positive integers ℓ and s, we have

$$\int_0^{\pi} [P_\ell^1(\cos\theta)(d/d\theta) P_s^1(\cos\theta) - P_s^1(\cos\theta)(d/d\theta) P_\ell^1(\cos\theta)] d\theta = 0, \tag{171}$$

it, thus, follows from equation 170 that

$$F_{(\ell,s)}^{(\beta,\alpha)} = 0 \, , \tag{172}$$

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and an almost identical argument shows that

$$F^{(\alpha,\beta)}_{(\ell,s)} = 0 \tag{173}$$

for all ℓ and s. Note that the value of W_s is independent of r. This enables us to make use of the asymptotic formulas

$$h^1_n(\rho) \cong \frac{1}{\rho}(-i)^{n+1}e^{i\rho} \tag{174}$$

and

$$(d/d\rho)h^1_n(\rho) \cong -\frac{1}{\rho^2}(-i)^{n+1}e^{i\rho} + \frac{i}{\rho}(-i)^{n+1}e^{i\rho} . \tag{175}$$

Further, we have

$$F^{(\alpha,\alpha)}_{(\ell,s)} = \int_0^{2\pi}\int_0^\pi \frac{k_N}{\mu_o\omega} E_0^2\, i^{s+\ell}(-1)^s[(M^{(o,3)}_{(1,\ell)})_\phi\, (-i)\overline{(N^{(o,3)}_{(1,s)})_\theta}$$

$$-(M^{(o,3)}_{(1,\ell)})_\theta\, (-i)\overline{(N^{(o,3)}_{(1,s)})_\phi}]r^2\sin\theta\,d\theta\,d\phi$$

$$= \int_0^\pi \frac{\pi k_N}{\mu_o\omega} E_0^2\, i^{s+\ell}(-1)^s(-i)[(M^{(o,3)}_{(1,\ell)})_\phi\, \overline{(N^{(o,3)}_{(1,s)})}$$

$$-(M^{(o,3)}_{(1,\ell)})_\theta\, \overline{(N^{(o,3)}_{(1,s)})_\phi}]r^2\sin\theta\,d\theta ;$$

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$$F^{(\alpha,\alpha)}_{(\ell,s)} = \int_0^\pi \frac{\pi k_N}{\mu_o \omega} E_o^2 \, i^{s+\ell} (-1)^s (-i) [h^1_\ell(k_N r)(1/k_N r)\overline{(\partial/\partial r)(r h^1_\ell(k_N r))}$$

$$-((d/d\theta)P^1_\ell(\cos\theta))((d/d\theta)P^1_s(\cos\theta)) - \frac{P^1_\ell(\cos\theta)P^1_s(\cos\theta)}{\sin^2\theta}] r^2 \sin\theta \, d\theta$$

$$= \frac{\pi k_N}{\mu_o \omega} E_o^2 \, i^{s+\ell+1}(-1)^s \frac{2}{2s+1}\frac{(s+1)!}{(s-1)!} s(s+1)\delta_{(s,\ell)}$$

$$\cdot h^1_\ell(k_N r)(1/k_N r)\overline{(\partial/\partial r)(r h^1_\ell(k_N r))} \tag{176}$$

after applying Proposition 6 for m=1.

To complete the calculation, we make use of the following lemma.

Lemma 1. For all k,

$$\lim_{r\to\infty} h^1_\ell(kr)(1/kr)\overline{(\partial/\partial r)(r h^1_\ell(kr))} r^2 = -i/k^2. \tag{177}$$

Proof. Equation 177 is equivalent to

$$\lim_{r\to\infty} h_\ell(kr)[\frac{r}{k}\overline{h^1_\ell(kr)} + r^2\overline{h^{1'}_\ell(kr)}] = -i/k^2 . \tag{178}$$

In view of equations 174 and 175, this completes the proof of Lemma 1. Using Lemma 1 and equation 176 we deduce that

$$F\binom{\alpha,\alpha}{\ell,s} = (\frac{\pi k_N}{\mu_o \omega}) \frac{E_o^2\delta_{(\ell,s)}i^{2s}(-1)^s i(-i)2s^2(s+1)^2}{K_N^2(2s+1)}$$

$$= (\frac{\pi k_N}{\mu_o \omega})2E_o^2\delta_{(\ell,s)}s^2(s+1)^2/((2s+1)K_N^2) \tag{179}$$

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Similarly

$$F^{(\beta,\beta)}_{(\ell,s)} = (\frac{\pi k_N}{\mu_o \omega})2E_o^2 \, \delta_{(\ell,s)}s^2(s+1)^2/((2s+1)k_N^2) \; . \tag{180}$$

Thus

$$W_s = (\frac{2\pi k_N}{\mu_o \omega})E_o^2 \sum_{s=1}^{\infty} (2s+1)(|\alpha_{(s,N)}|^2 + |\beta_{(s,N)}|^2)/k_N^2 \; . \tag{181}$$

Energy balance is maintained through the introduction of a third term, namely

$$W_t = -\text{Re} \int_o^{2\pi} \int_o^{\pi} (E_\theta^r \bar{H}_\phi^i + E_\theta^i \bar{H}_\phi^r - E_\phi^r \bar{H}_\theta^i - E_\phi^i \bar{H}_\theta^r)\sin\theta d\theta d\phi \; . \tag{182}$$

Thus, collecting multiples of the expansion coefficients, we obtain

$$W_t = -\text{Re}\left[\int_o^{2\pi} \int_o^{\pi} \{ \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)}(M^{(o,3)}_{(1,\ell)})_\theta - i\beta_{(\ell,N)}(N^{(e,3)}_{(1,\ell)})_\theta]\right.$$

$$\cdot \sum_{s=1}^{\infty} (-i)^s \frac{2s+1}{s(s+1)}[\overline{(M^{(e,1)}_{(1,s)})_\phi} - i\overline{(N^{(o,1)}_{(1,s)})_\phi}]$$

$$+ \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)}[(M^{(o,1)}_{(1,\ell)})_\theta - i(N^{(e,1)}_{(1,\ell)})_\theta]$$

$$\cdot \left\{ \sum_{s=1}^{\infty} i^s \, \frac{2s+1}{s(s+1)} [\bar{\beta}_{(s,N)} \overline{(M^{(e,3)}_{(1,s)})}_\phi - i\bar{\alpha}_{(s,N)} \overline{(N^{(o,3)}_{(1,s)})}_\phi] \right\}$$

$$- \sum_{\ell=1}^{\infty} i^\ell \, \frac{2\ell+1}{\ell(\ell+1)} [\alpha_{(\ell,N)} (M^{(o,3)}_{(1,\ell)})_\phi - i\beta_{(\ell,N)} (N^{(e,3)}_{(1,\ell)})_\phi]$$

$$\cdot \sum_{s=1}^{\infty} (-i)^s \, \frac{2s+1}{s(s+1)} [\overline{(M^{(e,1)}_{(1,\ell)})}_\theta - i \overline{(N^{(o,1)}_{(1,s)})}_\theta]$$

$$- \sum_{\ell=1}^{\infty} \sum_{s=1}^{\infty} [i^{\ell+s} (-1)^s \, \frac{(2\ell+1)(2s+1)}{\ell(\ell+1)s(s+1)} [(M^{(o,1)}_{(1,\ell)})_\phi - i(N^{(e,1)}_{(1,\ell)})_\phi]$$

$$\cdot [\bar{\beta}_{(s,N)} \overline{(M^{(e,3)}_{(1,s)})}_\theta - i\bar{\alpha}_{(s,N)} \overline{(N^{(o,3)}_{(1,s)})}_\theta]] \} \sin\theta \, d\theta \, d\phi \Bigg] \left(\frac{E_o^2 k_N}{\mu_o \omega} \right)$$

$$= - \,\text{Re} \left\{ \sum_{\ell=1}^{\infty} \sum_{s=1}^{\infty} i^{\ell+s} (-1)^s \, \frac{(2\ell+1)(2s+1)}{\ell(\ell+1)s(s+1)} [\alpha_{(\ell,N)} F^{(\alpha,1)}_{(\ell,s)} \right.$$

$$\left. + \bar{\alpha}_{(s,N)} F^{(1,\alpha)}_{(\ell,s)} + \beta_{(\ell,N)} F^{(\beta,1)}_{(\ell,s)} + \bar{\beta}_{(s,N)} F^{(1,\beta)}_{(\ell,s)}] \right\} \left(\frac{E_o^2 k_N}{\mu_o \omega} \right), \qquad (183)$$

where

$$F^{(\alpha,1)}_{(\ell,s)} = \int_S \{ (M^{(o,3)}_{(1,\ell)})_\theta [\overline{(M^{(e,1)}_{(1,s)})}_\phi - i(N^{(o,1)}_{(1,s)})_\phi]$$

$$- (M^{(o,3)}_{(1,\ell)})_\phi [\overline{(M^{(e,1)}_{(1,s)})}_\theta - i\overline{(N^{(o,1)}_{(1,s)})}_\theta] \} dA, \qquad (184)$$

$$F^{(1,\alpha)}_{(\ell,s)} = \int_S \{ [(M^{(o,1)}_{(1,\ell)})_\theta - i(N^{(e,1)}_{(1,\ell)})_\theta](-i)\overline{(N^{(o,3)}_{(1,s)})_\phi}$$

$$- [(M^{(o,1)}_{(1,\ell)})_\phi - i(N^{(e,1)}_{(1,\ell)})_\phi](-i)\overline{(N^{(o,3)}_{(1,s)})_\theta} \} dA, \qquad (185)$$

$$F^{(\beta,1)}_{(\ell,s)} = \int_S \{ (N^{(e,3)}_{(1,\ell)})_\theta (-i)[\overline{(M^{(e,1)}_{(1,s)})_\phi} - i\overline{(N^{(o,1)}_{(1,s)})_\phi}]$$

$$- (N^{(e,3)}_{(1,\ell)})_\phi (-i)[\overline{(M^{(e,1)}_{(1,s)})_\theta} - i\overline{(N^{(o,1)}_{(1,s)})_\theta}]\} dA, \qquad (186)$$

and

$$F^{(1,\beta)}_{(\ell,s)} = \int_S \{ \overline{(M^{(e,3)}_{(1,s)})_\phi} [(M^{(o,1)}_{(1,\ell)})_\theta - i(N^{(e,1)}_{(1,\ell)})_\theta]$$

$$- \overline{(M^{(e,3)}_{(1,s)})_\theta} [(M^{(o,1)}_{(1,\ell)})_\phi - i(N^{(e,1)}_{(1,\ell)})_\phi]\} dA. \qquad (187)$$

First we compute $F^{(\alpha,1)}_{(\ell,s)}$. Observe that if we let

$$A^{(\alpha,1)}_{(\ell,s)} = \int_0^{2\pi} \int_0^\pi \frac{1}{\sin\theta} h^1_\ell(k_N r) P^1_\ell(\cos\theta)\cos\phi$$

$$[- \overline{j_s(k_N r)}((d/d\theta)P^1_s(\cos\theta))\cos\phi$$

$$- \frac{i}{k_N r \sin\theta}(\partial/\partial r)\overline{(rj_s(k_N r))}P^1_s(\cos\theta)\cos\phi] r^2 \sin\theta\, d\theta\, d\phi$$

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$$A^{(\alpha,1)}_{(\ell,s)} = \pi r^2 h_\ell^1(k_N r)(-j_s(k_N r)) \int_0^\pi P_\ell^1(\cos\theta)((d/d\theta)P_s^1(\cos\theta))d\theta$$

$$- i\pi r^2 h_\ell^1(k_N r)(\frac{1}{k_N r})(\partial/\partial r)(\overline{rj_s(k_N r)}) \int_0^\pi \frac{P_\ell^1(\cos\theta)P_s^1(\cos\theta)}{\sin\theta} d\theta \quad (188)$$

and

$$B^{(\alpha,1)}_{(\ell,s)} = \int_0^{2\pi} \int_0^\pi - h_\ell^1(k_N r)((d/d\theta)P_\ell^1(\cos\theta))\sin\phi \,[$$

$$- \frac{1}{\sin\theta} \overline{j_s(k_N r)}P_s^1(\cos\theta)\sin\phi$$

$$- \frac{i}{k_N r} (\partial/\partial r)(\overline{rj_s(k_N r)})((d/d\theta)P_s^1(\cos\theta))\sin\phi]r^2\sin\theta d\theta$$

$$= \pi r^2 h_\ell^1(k_N r)\overline{j_s(k_N r)}\Big|_0^\pi \int_0^\pi ((d/d\theta)P_\ell^1(\cos\theta))P_s^1(\cos\theta)d\theta$$

$$+ \frac{\pi i r}{k_N} h_\ell^1(k_N r)(\partial/\partial r)(\overline{rj_s(k_N r)}) \int_0^\pi [(d/d\theta)P_\ell^1(\cos\theta)$$

$$\cdot (d/d\theta)P_s^1(\cos\theta)]\sin\theta d\theta, \quad (189)$$

then

$$F^{(\alpha,1)}_{(\ell,s)} = A^{(\alpha,1)}_{(\ell,s)} - B^{(\alpha,1)}_{(\ell,s)} . \quad (190)$$

In view of equations 18 and 170, we obtain

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$$F^{(\alpha,1)}_{(\ell,s)} = A^{(\alpha,1)}_{(\ell,s)} - B^{(\alpha,1)}_{(\ell,s)}$$

$$= -\pi r^2 h^1_\ell(k_N r)\overline{j_s(k_N r)} \int_0^\pi [P^1_\ell(\cos\theta)((d/d\theta)P^1_s(\cos\theta))$$

$$+ P^1_s(\cos\theta)((d/d\theta)P^1_\ell(\cos\theta))]\,d\theta$$

$$- \frac{i\pi r^2 h^1_\ell(k_N r)(\partial/\partial r)\overline{(rj_s(k_N r))}}{k_N r} \int_0^\pi [-\frac{P^1_\ell(\cos\theta)P^1_s(\cos\theta)}{\sin^2\theta}$$

$$+ ((d/d\theta)P^1_\ell(\cos\theta))((d/d\theta)P^1_s(\cos\theta))]\sin\theta\,d\theta$$

$$= - \frac{i\pi r^2 h^1_\ell(k_N r)(\partial/\partial r)\overline{(rj_s(k_N r))}}{k_N r}\delta_{(\ell,s)}$$

$$\cdot \frac{2}{2s+1}\frac{(s+1)!}{(s-1)!}\,s(s+1)$$

$$= - \frac{i\pi r h^1_\ell(k_N r)(\partial/\partial r)\overline{(rj_s(k_N r))}}{k_N}\delta_{(\ell,s)}\frac{2s^2(s+1)^2}{2s+1}\,. \tag{191}$$

Next we compute $F^{(1,\alpha)}_{(\ell,s)}$. As before, we let

$$A^{(1,\alpha)}_{(\ell,s)} = -i \int_S [(M^{(o,1)}_{(1,\ell)})_\theta -i(N^{(e,1)}_{(1,\ell)})_\theta]\overline{(N^{(o,3)}_{(1,s)})}\,dA \tag{192}$$

and

$$B^{(1,\alpha)}_{(\ell,s)} = -i \int_S [(M^{(o,1)}_{(1,\ell)})_\phi -i(N^{(e,1)}_{(1,\ell)})_\phi]\overline{(N^{(o,3)}_{(1,s)})_\theta}\,dA. \tag{193}$$

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Use of equations 153-156 yields

$$A^{(1,\alpha)}_{(\ell,s)} = -i \int_0^{2\pi} \int_0^{\pi} [\frac{1}{\sin\theta} j_\ell(k_N r)P^1_\ell(\cos\theta)\cos\phi$$

$$- i\frac{1}{k_N r} (\partial/\partial r)(rj_\ell(k_N r))(d/d\theta)P^1_\ell(\cos\theta)\cos\phi]$$

$$\cdot \frac{1}{k_N r\sin\theta} (\partial/\partial r)(\overline{rh^1_s(k_N r)})P^1_s(\cos\theta)\cos\phi r^2\sin\theta d\theta d\phi$$

$$= - \frac{i\pi r}{k_N} j_\ell(k_N r)(\partial/\partial r)(\overline{rh^1_s(k_N r)}) \int_0^{\pi} \frac{P^1_\ell(\cos\theta)P^1_s(\cos\theta)}{\sin\theta} d\theta$$

$$- \frac{i\pi}{k_N^2} ((\partial/\partial r)(rj_\ell(k_N r)))((\partial/\partial r)(\overline{rh^1_s(k_N r)}))$$

$$\cdot \int_0^{\pi} ((d/d\theta)P^1_\ell(\cos\theta))P^1_s(\cos\theta)d\theta . \tag{194}$$

Also, using equations 153-156 and 193, we deduce that

$$B^{(1,\alpha)}_{(\ell,s)} = -i \int_0^{2\pi} \int_0^{\pi} [(-j_\ell(k_N r))(d/d\theta)P^1_\ell(\cos\theta)\sin\phi$$

$$+ \frac{i}{k_N r\sin\theta} (\partial/\partial r)(rj_\ell(k_N r))P^1_\ell(\cos\theta)\sin\phi]$$

$$\cdot [\frac{1}{k_N r} (\partial/\partial r)(\overline{rh^1_s(k_N r)})(d/d\theta)P^1_s(\cos\theta)\sin\phi]r^2\sin\theta d\theta d\phi .$$

Carrying out the integration with respect to ϕ we see that

$$B_{(\ell,s)}^{(1,\alpha)} = \frac{i\pi r}{k_N} j_\ell(k_N r)(\partial/\partial r)\overline{(rh_s^1(k_N r))}$$

$$\cdot \int_0^\pi ((d/d\theta)P_\ell^1(\cos\theta))((d/d\theta)P_s^1(\cos\theta))\sin\theta d\theta$$

$$+ \frac{i\pi}{k_N^2}((\partial/\partial r)(rj_\ell(k_N r)))(\partial/\partial r)\overline{(rh_s^1(k_N r))}$$

$$\cdot \int_0^\pi P_\ell^1(\cos\theta)(d/d\theta)P_s^1(\cos\theta)d\theta. \tag{195}$$

From equations 185, 194, and 195, it follows that

$$F_{(\ell,s)}^{(1,\alpha)} = A_{(\ell,s)}^{(1,\alpha)} - B_{(\ell,s)}^{(1,\alpha)}. \tag{196}$$

Thus,

$$F_{(\ell,s)}^{(1,\alpha)} = -\frac{i\pi r}{k_N} j_\ell(k_N r)(\partial/\partial r)\overline{(rh_s^1(k_N r))}$$

$$\cdot \int_0^\pi [\frac{P_\ell^1(\cos\theta)P_s^1(\cos\theta)}{\sin^2\theta} + ((d/d\theta)P_\ell^1(\cos\theta))((d/d\theta)P_s^1(\cos\theta))]\sin\theta d\theta$$

$$- \frac{i\pi}{k_N^2}((\partial/\partial r)(rj_\ell(k_N r)))((\partial/\partial r)\overline{(rh_s^1(k_N r))})$$

$$\cdot \int_0^\pi [((d/d\theta)(P_\ell^1(\cos\theta)))P_s^1(\cos\theta) + ((d/d\theta)P_s^1(\cos\theta))P_\ell^1(\cos\theta)]d\theta$$

$$= -\frac{i\pi r}{k_N} j_\ell(k_N r)(\partial/\partial r)\overline{(rh_s^1(k_N r))}[\delta_{(\ell,s)}\frac{2s^2(s+1)^2}{2s+1}]. \tag{197}$$

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Furthermore

$$\text{Re}(\alpha_{(s,N)} F^{(\alpha,1)}_{(s,s)} + \bar{\alpha}_{(s,N)} F^{(1,\alpha)}_{(s,s)})$$

$$= \text{Re}[\alpha_{(s,N)} \frac{-\pi r h^1_s(k_N r)(\partial/\partial r)(\overline{r j_s(k_N r)})}{k_N} \delta_{(\ell,s)} \frac{2s^2(s+1)^2}{2s+1}$$

$$+ \bar{\alpha}_{(s,N)} \frac{-i\pi r j_s(k_N r)(\partial/\partial r)(\overline{r h^1_s(k_N r)})}{k_N} \delta_{(\ell,s)} \frac{2s^2(s+1)^2}{2s+1}]$$

$$= \text{Re}(\alpha_{(s,N)}) \frac{2s^2(s+1)^2}{2s+1} \quad . \tag{198}$$

Hence, using the fact that k_N is real, we have

$$W_t = \frac{2\pi E_o^2}{k_N^2} \sqrt{\varepsilon_o/\mu_o} \text{Re} \sum_{s=1}^{\infty} (2s+1)(\alpha_{(s,N)} + \beta_{(s,N)}) \quad . \tag{199}$$

This follows from an induction argument and the fact that if

$$u + iv = \frac{(-i)^{n+1}}{k_N} [\cos(k_N r) + i \sin(k_N r)] \tag{200}$$

and

$$w = \frac{1}{k_N} \cos[k_N r - (\frac{n+1}{2})\pi] \quad , \tag{201}$$

then for all real numbers A and B,

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$$\text{Re}\{i[w'(u+iv)(A+iB) + w(u'-iv')(A-iB)]\}$$

$$= (v'w-vw')A + (u'w-uw')B$$

$$= \frac{\Lambda}{k_N} \tag{202}$$

for every positive integer n. A prime on u, v, or w denotes differentiation with respect to r.

Thus, time averaging shows that the total absorbed power is given by

$$W_a = \left| \frac{\pi E_o^2}{k_N^2}\sqrt{\varepsilon_o/\mu_o} \ \text{Re} \sum_{n=1}^{\infty} (2n+1)(\alpha_{(n,N)} + \beta_{(n,N)}) \right|$$

$$- \frac{\pi E_o^2}{k_N^2}\sqrt{\varepsilon_o/\mu_o} \sum_{n=1}^{\infty} (2n+1)(|\alpha_{(n,N)}|^2 + |\beta_{(n,N)}|^2). \tag{203}$$

Summary of Key Equations and Formulas

In summarizing, we set down the key equations and formulas upon which program CSM is based.

Fields for the p-th region:

$$E_p = E_o \exp(-\omega t) \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [a_{(\ell,p)}\vec{M}_{(1,\ell)}^{(o,1)} - ib_{(\ell,p)}\vec{N}_{(1,\ell)}^{(e,1)}$$

$$+ \alpha_{(\ell,p)}\vec{M}_{(1,\ell)}^{(o,3)} - i\beta_{(\ell,p)}\vec{N}_{(1,\ell)}^{(e,3)}], \tag{204}$$

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$$H_p = -\frac{k_p}{\mu_o\omega} E_o \exp(-\omega t) \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)} [b_{(\ell,p)}\vec{M}^{(e,1)}_{(1,\ell)} + ia_{(\ell,p)}\vec{N}^{(o,1)}_{(1,\ell)}$$

$$+ \beta_{(\ell,p)}\vec{M}^{(e,3)}_{(1,\ell)} + i\alpha_{(\ell,p)}\vec{N}^{(o,3)}_{(1,\ell)}] \tag{205}$$

where the vector wave functions $\vec{M}^{(e,3)}_{(1,\ell)}$, $\vec{M}^{(o,3)}_{(1,\ell)}$, $\vec{N}^{(e,3)}_{(1,\ell)}$, and $\vec{N}^{(o,3)}_{(1,\ell)}$ are obtained by replacing the spherical Bessel function $j_n(k_p r)$ by the spherical Hankel function $h_n^{(1)}(k_p r)$ in the expressions for the vector wave functions $\vec{M}^{(e,1)}_{(1,\ell)}$, $\vec{M}^{(o,1)}_{(1,\ell)}$, $\vec{N}^{(e,1)}_{(1,\ell)}$, and $\vec{N}^{(o,1)}_{(1,\ell)}$.

Complex propagation constant for the p-th region:

$$k_p = \mathrm{Re}(k_p) + i\,\mathrm{Im}(k_p), \tag{206}$$

where

$$\mathrm{Re}(k_p) = \frac{\omega}{c} \left\{ \frac{\epsilon_p}{2} \left[\left(1 + \frac{1}{(\epsilon_o\omega)^2}(\frac{\sigma_p}{\epsilon_p})^2\right)^{\frac{1}{2}} + 1 \right] \right\}^{\frac{1}{2}}, \tag{207}$$

$$\mathrm{Im}(k_p) = \frac{\omega}{c} \left\{ \frac{\epsilon_p}{2} \left[\left(1 + \frac{1}{(\epsilon_o\omega)^2}(\frac{\sigma_p}{\epsilon_p})^2\right)^{\frac{1}{2}} - 1 \right] \right\}^{\frac{1}{2}}, \tag{208}$$

ϵ_0 = free-space permittivity; = 8.85×10^{-12} F/m,

ϵ_p = relative dielectric constant of p-th region; = 1 for free space,

σ_p = conductivity of the p-th region; = 0 for free space,

ω = angular frequency; = $2\pi \times$ frequency (in MHz),

c = velocity of light in free space; = 2.9979×10^8 m .

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The field expansion coefficients for region one, inner core sphere, and those for the surrounding medium are obtained through the solution of two systems of equations. Utilizing the notation of Shapiro et al. (13), we have, with $a_{(1,N)} = b_{(1,N)} = 1$ and $\alpha_{(1,1)} = \beta_{(1,1)} = 0$,

$$\begin{bmatrix} \alpha_{(\ell,1)} \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{\ell,T}^{ij} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_{(\ell,N)} \end{bmatrix} , \tag{209}$$

$$\begin{bmatrix} b_{(\ell,1)} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{\ell,T}^{ij} \end{bmatrix} \begin{bmatrix} 1 \\ \beta_{(\ell,N)} \end{bmatrix} , \tag{210}$$

where the product matrices $[Q_{1T}^{ij}]$ and $[R_{1T}^{ij}]$ have the representation

$$\begin{bmatrix} Q_{\ell,T}^{ij} \end{bmatrix} = \prod_{p=1}^{N-1} \begin{bmatrix} Q_{(i,j)}^{(\ell,p)} \end{bmatrix} , \tag{211}$$

$$\begin{bmatrix} R_{\ell,T}^{ij} \end{bmatrix} = \prod_{p=1}^{N-1} \begin{bmatrix} R_{(i,j)}^{(\ell,p)} \end{bmatrix} , \tag{212}$$

with each factor matrix $[Q_{(i,j)}^{(\ell,p)}]$ and $[R_{(i,j)}^{(\ell,p)}]$ having its (i,j) elements computed by means of the following formulas:

$$Q_{(1,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\xi_{(\ell,p)}^{+} j_{(\ell,p+1)}^{-} - \frac{k_{p+1}}{k_p} h_{(\ell,p)}^{+} n_{(\ell,p+1)}^{-}] , \tag{213}$$

$$Q_{(1,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\xi_{(\ell,p)}^{+} h_{(\ell,p+1)}^{-} - \frac{k_{p+1}}{k_p} h_{(\ell,p)}^{+} \xi_{(\ell,p+1)}^{-}] , \tag{214}$$

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$$Q_{(2,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\frac{k_{p+1}}{k_p} j_{(\ell,p)}^+ n_{(\ell,p+1)}^- - n_{(\ell,p)}^+ j_{(\ell,p+1)}^-] \quad , \tag{215}$$

$$Q_{(2,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\frac{k_{p+1}}{k_p} j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - n_{(\ell,p)}^+ h_{(\ell,p+1)}^-] \quad , \tag{216}$$

$$R_{(1,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\frac{k_{p+1}}{k_p} \xi_{(\ell,p)}^+ j_{(\ell,p+1)}^- - h_{(\ell,p)}^+ n_{(\ell,p+1)}^-] \quad , \tag{217}$$

$$R_{(1,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[\frac{k_{p+1}}{k_p} \xi_{(\ell,p)}^+ h_{(\ell,p+1)}^- - h_{(\ell,p)}^+ \xi_{(\ell,p+1)}^-] \quad , \tag{218}$$

$$R_{(2,1)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[j_{(\ell,p)}^+ n_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} n_{(\ell,p)}^+ j_{(\ell,p+1)}^-] \quad , \tag{219}$$

$$R_{(2,2)}^{(\ell,p)} = (\Delta_{(\ell,p)})^{-1}[j_{(\ell,p)}^+ \xi_{(\ell,p+1)}^- - \frac{k_{p+1}}{k_p} n_{(\ell,p)}^+ h_{(\ell,p+1)}^-] \quad , \tag{220}$$

Here

$$\Delta_{(\ell,p)} = j_{(\ell,p)}^+ \xi_{(\ell,p)}^+ - h_{(\ell,p)}^+ n_{(\ell,p)}^+ \quad , \tag{221}$$

$$j_{(\ell,p)}^+ = j_\ell(k_p r) \quad , \tag{222}$$

$$h_{(\ell,p)}^+ = h_\ell(k_p r) \quad , \tag{223}$$

$$n_{(\ell,p)}^+ = \frac{1}{2\ell+1} [(\ell+1)j_{(\ell-1,p)}^+ - \ell j_{(\ell+1,p)}^+] \quad , \tag{224}$$

$$\xi_{(\ell,p)}^+ = \frac{1}{2\ell+1} [(\ell+1)h_{(\ell-1,p)}^+ - \ell h_{(\ell+1,p)}^+] \quad , \tag{225}$$

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the superscript 1 has been dropped from the spherical Hankel functions $h_n^{(1)}(k_p r) = j_n(k_p r) + i y_n(k_p r)$, and r is the radius of the boundary surface of the p-th region for subscripts p and $p+1$.

The matrix equations

$$\begin{bmatrix} a_{(\ell,p)} \\ \alpha_{(\ell,p)} \end{bmatrix} = \begin{bmatrix} Q\begin{pmatrix} \ell,p \\ i,j \end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{(\ell,p+1)} \\ \alpha_{(\ell,p+1)} \end{bmatrix} \quad , \tag{226}$$

$$\begin{bmatrix} b_{(\ell,p)} \\ \beta_{(\ell,p)} \end{bmatrix} = \begin{bmatrix} R\begin{pmatrix} \ell,p \\ i,j \end{pmatrix} \end{bmatrix} \begin{bmatrix} b_{(\ell,p+1)} \\ \beta_{(\ell,p+1)} \end{bmatrix} \quad , \tag{227}$$

yield the expansion coefficients for the regions $p = 2,\ldots,N-1$ in a recursive manner, starting with derived values of $a_{(1,1)}$ and $b_{(1,1)}$ and known values of $\alpha_{(1,1)}$ and $\beta_{(1,1)}$ as elements in the left-hand members of the matrix equations, and employing equations 213-225 for computing the necessary coefficient matrics $[Q_{(i,j)}^{(\ell,p)}]$ and $[R_{(i,j)}^{(\ell,p)}]$.

Absorbed-power density at an interior point of the p-th region:

$$P = 0.5\sigma_p(\vec{E}_p \cdot \vec{E}_p^*) \quad , \tag{228}$$

where

$\vec{E}_p$ = electric vector at an interior point of the p-th region,

σ_p = conductivity of the p-th region,

$*$ = complex conjugate indicator.

Average absorbed-power density:

$$P_{avg} = (3/8\pi)(\epsilon_0/\mu_0)^{\frac{1}{2}}(E_0^2 Q_a/r_{N-1}^3), \tag{229}$$

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where

$$Q_a = |\frac{2\pi}{k_N^2} \text{Re} \sum_{\ell=1}^{\infty} (2\ell+1)(\alpha_{\ell,N} + \beta_{\ell,N})| - \frac{2\pi}{k_N^2} \sum_{\ell=1}^{\infty} (2\ell+1)(|\alpha_{\ell,N}|^2$$

$$+ |\beta_{\ell,N}|^2) = Q_t - Q_s , \tag{230}$$

ε_0, μ_0 = free-space permittivity and permeability ,

k_N = propagation constant of the surrounding medium,

$\alpha_{\ell,N}, \beta_{\ell,N}$ = scattering coefficients.

Total absorbed power:

$$P_{tot} = \frac{2P_i}{\alpha^2} \sum_{\ell=1}^{\infty} (2\ell+1)[|\text{Re}(\alpha_{\ell,N} + \beta_{\ell,N})| - (|\alpha_{\ell,N}|^2 + |\beta_{\ell,N}|^2)] , \tag{231}$$

where

P_i = power incident upon α; $= (\frac{E_0^2}{2n})\pi r_{N-1}^2$,

n = intrinsic impedance for free space; = 376.7 ohms,

r_{N-1} = radius of spherical surface adjacent to the surrounding medium,

α = geometrical cross section of the sphere of radius r_{N-1}; $= 2\pi r_{N-1}/\lambda$,

λ = wavelength of the incident wave.

To complete our summarization, consideration of the formulas used in generating the values of certain functions seems appropriate. The formulas

$$P_{n+1}^1(\cos\theta) = \frac{2n+1}{n} \cos\theta \, P_n^1(\cos\theta) - \frac{n+1}{n} P_{n-1}^1(\cos\theta) , \tag{232}$$

$$\sin\theta(d/d\theta)P_n^1(\cos\theta) = n\cos\theta P_n^1(\cos\theta) - (n+1)P_{n-1}^1(\cos\theta) , \tag{233}$$

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together with

$$P_1^1(\cos\theta) = \sin\theta, \qquad (234)$$

$$P_2^1(\cos\theta) = 3 \sin\theta\cos\theta \qquad (235)$$

are used to generate function and derivative values of the associated Legendre functions.

Special limit values are also obtained by

$$\lim_{\theta\to 0} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{n(n+1)}{2} , \qquad (236)$$

$$\lim_{\theta\to\pi} \frac{P_n^1(\cos\theta)}{\sin\theta} = \frac{(-1)^{n+1}n(n+1)}{2} . \qquad (237)$$

The forward recurrence relation

$$y_{n+1}(z) + y_{n-1}(z) = \frac{2n+1}{z} y_n(z) \qquad (238)$$

is used together with relations

$$y_0(z) = - \frac{\cos z}{z} , \qquad (239)$$

$$y_1(z) = - \frac{\cos z}{z^2} - \frac{\sin z}{z} \qquad (240)$$

to generate values of the spherical Neuman functions. The generating process is terminated at order N when the following termination criterion

$$\left| y_n(z) \right| \geq \text{STOPR} \qquad (241)$$

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is met. Here STOPR is a number, say 1.0D15. The user's needs will determine whether or not STOPR should retain its presently suggested value. Our own demands were satisfactorily met for complex argument, $\epsilon_p r$, of the spherical Neumann functions for parameter ranges:

$1.5 \leq |\epsilon_p| \leq 1390.0$ and $0.1 \leq r \leq 10$ cm.

The backward recurrence relation

$$j_{n-1}(z) = \frac{2n+1}{z} j_n(z) - j_{n+1}(z) \tag{242}$$

in combination with an appropriate starting value is used to generate values of the spherical Bessel functions of the first kind, $j_n(z)$. This technique of using the backward relation in place of the forward relation helps to avoid stability problems.

PROGRAM DESCRIPTION

Written in standard FORTRAN IV for the IBM 360/65 system, the Concentric Spherical Model (CSM) is designed to calculate the internal absorbed-power density distributions, average absorbed-power density, and total absorbed power for a spherical shell configuration (simulating the human head) subjected to plane-wave, nonionizing electromagnetic radiation. Five spherical shells plus a brain core sphere are generally treated, but provision is made to allow as many as eight concentric shells to be analyzed. The structural components of the head model are identified by regional designators. Regions 1 through 6 represent the brain, cerebrospinal fluid (CSF), dura, bone, subcutaneous fat, and skin, respectively. Current plotting needs of the USAF School of Aerospace Medicine are met through the use of BGNSTP--an advanced general plotting subroutine package--and the use of the CalComp Model 936 digital incremental plotter. Since the plotting package is not available for distribution, the plotter calls and working arrays are not included in this published version of the program.

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Basically, program CSM consists of a driver routine, five sub-routine subprograms, and one function subprogram. These routines are single-entry programs, free of any special machine dependence, and utilize subprograms found in any elementary software library. Types REAL*8 and COMPLEX*16 signify double-precision real and complex variables, respectively, and literal data appearing in a FORMAT state-ment are enclosed in apostrophes. The arithmetical processes are per-formed in double-precision, floating-point mode. This feature provides approximately 16.8 decimal digits of precision and numbers with an exponent range of -78 to +75. A list of the driver routine and sub-programs, including function, calling sequence, and calling arguments of each member, follows.

Driver routine:

Routine MAIN is used to input/output data; to compute complex propagation constants; to complete the calculations for the absorbed-power density distributions, average absorbed-power density, and total absorbed power; to control the printing activities; and to direct the course of calculations.

Subroutine subprograms:

Subroutine COEF generates the expansion coefficients for the com-ponents of the electric-field vectors $\vec{E}_p$, p = 1,...,NOREG.

The calling sequence of this subroutine is

COEF(ANP, BNP, ALPNP, BETNP, NMIN)

where the calling arguments are

ANP = array of coefficients for vector functions $\vec{M}^{(o,1)}_{(1,n)}$,

BNP = array of coefficients for vector functions $\vec{N}^{(e,1)}_{(1,n)}$,

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ALPNP = array of coefficients for vector functions $\vec{M}^{(o,3)}_{(1,n)}$,

BETNP = array of coefficients for vector functions $\vec{N}^{(e,3)}_{(1,n)}$,

NMIN = number of terms in the series expansion of each component of the electric-field vector, $\vec{E}_p$.

The above arrays are double-precision, complex, and each array is dimensioned at 1000.

Subroutine EVEC computes the radial, colatitude, and azimuthal components and the scalar product $\vec{E}_p \cdot \vec{E}_p^*$ for the electric-field vectors $\vec{E}_p$, $p = 1,\ldots,$NOREG.

The calling sequence of this subroutine is

EVEC(NP,PD)

where the calling arguments are

NP = region identifier,

PD = double-precision, complex, semicompleted absorbed-power density at an internal point of the p-th region.

Subroutine TERM computes $(-1)^n$ or $(-1)^{n+1}$ times the appropriate part of the n-th term in the series expansion of each component of the electric-field vectors $\vec{E}_p$, $p = 1,\ldots,$NOREG.

The calling sequence of this subroutine is

TERM(NCK, T, KEY)

where the calling arguments are

NCK = a counter count,

T   = part of the n-th term in the series expansion
       that is multiplied by the appropriate power
       of -1,

KEY = 0 for T to be multiplied by $(-1)^n$ and 1
       for T to be multiplied by $(-1)^{n+1}$.

The array T is double-precision, complex.

Subroutine BJYH generates the spherical Bessel functions $j_n(k_p r)$,
spherical Neumann functions $y_n(k_p r)$, and spherical Hankel functions
$h_n^{(1)}(k_p r)$.

The calling sequence of this subroutine is

BJYH(BJNP, BHNP, Z, NN, STOPR)

where the calling arguments are

BJNP = array of spherical Bessel functions for
        p-th region,

BHNP = array of spherical Hankel functions for the
        p-th region,

Z   = product of complex propagation constant and
        radius of an internal point or boundary sur-
        face of the p-th region,

NN   = maximum order of the spherical functions,

STOPR = a test quantity for terminating the gener-
        ation of the spherical Neumann functions.

The arrays BJNP and BHNP are double-precision, complex, and
each array is dimensioned at 100. Variable Z is double-precision,
complex.

Subroutine PL generates the associated Legendre functions $P_n^1(\cos\theta)$ and their first derivatives with respect to θ.

The calling sequence for this subroutine is

PL(THETA, N, P, DP)

where the calling arguments are

THETA = value of the colatitude angle expressed in radians,

N = number of associated Legendre functions to be generated, starting with the function of degree one,

P = array of values of the associated Legendre functions,

DP = array of values of the first derivative of the associated Legendre functions.

The arrays P and DP are double-precision, real and are dimensioned at 101 and 100, respectively. THETA is a double-precision, real variable.

Function subprogram MINN determines the minimum value of a given array of positive integers.

The calling sequence for this function subprogram is

MINN(NRAY,N)

where the calling arguments are

NRAY = array of positive integers,

N = number of integers.

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The array NRAY is single-precision, integer, and dimensioned at 10.

Blank COMMON is used by the driver routine, MAIN, and the subroutines COEF and EVEC. The list of the arrays and variables stored in this area is

FKP* = wave propagation constants, k_p ,

BJNP* = spherical Bessel functions, $j_n(k_p r)$,

BHNP* = spherical Hankel functions, $h_n^{(1)}(k_p r)$,

CEX* = exponential value, exp(-iωt), for circular frequency ω and time t,

BDP = spherical surface boundaries,

P = associated Legendre functions, $P_n^1(\cos\theta)$,

DP = first derivative of the associated Legendre functions, $\frac{d}{d\theta} P_n^1(\cos\theta)$,

SIGP = conductivities, σ_p,

EO = intensity of the incident electric field, $\vec{E}$,

TIME = time,

R = radius of an internal point of the brain core sphere, or spherical shell region, or the radius of a spherical boundary surface,

THETA = colatitude angle, θ,

PHI = azimuthal angle, φ,

STOPR = a test quantity for terminating the generation of the spherical Neumann functions, $y_n(k_p r)$,

NC = maximum order of the spherical functions minus 2,

The array NRAY is single-precision, integer, and dimensioned at 10.

Blank COMMON is used by the driver routine, MAIN, and the subroutines COEF and EVEC. The list of the arrays and variables stored in this area is

FKP^* = wave propagation constants, k_p ,

$BJNP^*$ = spherical Bessel functions, $j_n(k_p r)$,

$BHNP^*$ = spherical Hankel functions, $h_n^{(1)}(k_p r)$,

CEX^* = exponential value, $\exp(-i\omega t)$, for circular frequency ω and time t,

BDP = spherical surface boundaries,

P = associated Legendre functions, $P_n^1(\cos\theta)$,

DP = first derivative of the associated Legendre functions, $\frac{d}{d\theta} P_n^1(\cos\theta)$,

SIGP = conductivities, σ_p,

EO = intensity of the incident electric field, $\vec{E}$,

TIME = time,

R = radius of an internal point of the brain core sphere, or spherical shell region, or the radius of a spherical boundary surface,

THETA = colatitude angle, θ,

PHI = azimuthal angle, ϕ,

STOPR = a test quantity for terminating the generation of the spherical Neumann functions, $y_n(k_p r)$,

NC = maximum order of the spherical functions minus 2,

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NOREG = number of regions in the Concentric Spherical Model,

NMIN = number of terms in the series expansions of the components of the electric-field vector, $\vec{E}_p$.

The double-precision, complex arrays and variables are flagged with an asterisk (*); while the unflagged arrays and variables are double-precision, real--with the exception of the last three members, of type INTEGER, which are single-precision variables.

A single-labeled common area, COEF, is used by the driver routine, MAIN, and the subroutine EVEC, for values of the expansion coefficients $a_{(n,p)}$, $b_{(n,p)}$, $\alpha_{(n,p)}$, $\beta_{(n,p)}$ stored in the arrays ANP, BNP, ALPNP, and BETNP respectively.

In subroutine BJYH, if variable M, the maximum order of the spherical Neumann functions $y_n(k_p r)$ (complex $k_p r$), tests ≤ 2, the error message

PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z = ...

is printed out and the computer run is terminated.

Input to program CSM is by keypunched cards. There are two basic input cards with structure and sequential order as follows:

Card No. 1 (control parameters)

Columns: 1-10 FREQ. Frequency in MHz. (E10.3)

11-20 EO. Intensity (field strength) of the incident electric field in volt/meter. (E10.3)

21-30 TIME. Time in seconds. (E10.3)

31-40 STOPR. A test quantity for terminating the generation of the spherical Neumann functions. A suggested value is 1.0E15. (E10.3)

41-45 NORG. Number of regions in the concentric spherical model of the human or animal head. (I5)

46-50 NOCR. Number of cases. (I5)

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Card No. 2 (electrical property data)

    Columns: 1-10 EPSP(1). Relative dielectric constant for
                     region 1. (E10.3)

              11-20 SIGP(1). Conductivity for region 1 in
                     mho/meter. (E10.3)

              21-30 EPSP(2). Relative dielectric constant for
                     region 2. (E10.3)

              31-40 SIGP(2). Conductivity for region 2 in
                     mho/meter. (E10.3)

              41-50 .   .   .

              51-60 .   .   .

              61-70 EPSP(4). Relative dielectric constant for
                     region 4. (E10.3)

              71-80 SIGP(4). Conductivity for region 4 in
                     mho/meter. (E10.3)

Card 3 is a similarly structured card for the electrical properties of
regions 5 and 6.

Card No. 4 (surface boundary data)

    Columns: 1-10 SBDP(1). Radius of the spherical surface for
                     region 1 in centimeters. (E10.3)

              11-20 SBDP(2). Radius of the spherical surface for
                     region 2 in centimeters. (E10.3)

              21-30 .   .   .

              31-40 .   .   .

              41-50 .   .   .

              51-60 SBDP(6). Radius of the spherical surface for
                     region 6 in centimeters. (E10.3)

Card Nos. 5-(NOCR+4) (coordinate data)

    Columns: 1-5   NREG. Region number. (I5)

              6-15 R. Radial spherical coordinate of an interior
                     point of region NREG in centimeters.
                     Range: $0 < R \leq$ SBDP(6). (E10.3)

16-25 THETAD. Colatitude spherical coordinate of an interior point of region NREG in degrees. Range: $0 \leq$ THETAD ≤ 180. (E10.3)

26-35 PHID. Azimuthal spherical coordinate of an interior point of region NREG in degrees. Range: $0 \leq$ PHID ≤ 360. (E10.3)

The last card of a single data set must be a termination card with the symbols /* punched in columns 1 and 2. Also program CSM can handle multiple data sets. Each data set [Cards 1-(NOCR+4)] is stacked one behind the other, with the last card in the complete data deck a termination card.

Program printouts consist of the title

ELECTROMAGNETIC ENERGY DEPOSTION IN A CONCENTRIC SPHERICAL MODEL

OF THE HUMAN OR ANIMAL HEAD

followed by such information as

FREQUENCY = ... MHZ

FIELD STRENGTH = . . . V/M

TIME = . . . SEC

NUMBER OF REGIONS = . . .

RELATIVE DIELECTRIC CONSTANTS = . . .

CONDUCTIVITIES (MHO/M) = . . .

SURFACE BOUNDARIES (CM) = . . .

REGION . . . INTERIOR POINT: RADIUS = . . . CM THETA = . . . DEG

PHI = . . . DEG ABSORBED-POWER DENSITY = . . . W/M**3

REGION . . . INTERIOR POINT: RADIUS = . . . CM THETA = . . . DEG

PHI = . . . DEG ABSORBED-POWER DENSITY = . . . W/M**3

. . .

. . .

AVERAGE ABSORBED-POWER DENSITY = . . . W/M**3

TOTAL ABSORBED POWER = ... WATT

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APPENDIX A

SAMPLE PROBLEM WITH COMPUTER RESULTS

SAMPLE PROBLEM DECK SETUP

CARD 1 (CONTROL PARAMETERS: FREQ,EO,TIME,STOPF,NOPG,NOCF)
1000.0+0   1.0+0   0.0+0   1.0+15   6   20

CARDS 2-3 (ELECTRICAL PROPERTIES: EPSP(4),SIGP(I))
60.0+0   0.1+0   76.0+0   1.7+0   45.0+0   1.0+0   8.5+0   0.11+0
5.5+0   0.08+0   45.0+0   1.0+0

CARD 4 (RADII OF BOUNDARY SURFACES: SBDF(4))
5.27+0   5.4+0   5.52+0   5.80+0   5.90+0   6.00+0

CARDS 5-24 (INTERNAL POINTS: NPEG,F,THETAI,PHID)
1   1.00-3   180.0+0   0.0+0
1   0.25+0   170.0+0   0.0+0
1   0.50+0   160.0+0   0.0+0
1   0.75+0   150.0+0   0.0+0
1   1.00+0   145.0+0   0.0+0
1   1.25+0   140.0+0   0.0+0
1   1.50+0   130.0+0   0.0+0
1   1.75+0   110.0+0   0.0+0
1   2.00+0   110.0+0   0.0+0
1   2.25+0   100.0+0   0.0+0
1   2.50+0   90.0+0   0.0+0
1   2.75+0   80.0+0   0.0+0
1   3.0+0   70.0+0   0.0+0
1   5.27+0   60.0+0   0.0+0
1   5.47+0   50.0+0   0.0+0
4   5.52+0   40.0+0   0.0+0
3   5.60+0   30.0+0   0.0+0
3   5.80+0   20.0+0   0.0+0
6   5.90+0   10.0+0   0.0+0
6   6.00+0   0.0+0   0.0+0

TERMINATION CARD
/*

ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD

FREQUENCY = 1000.00 MHZ   FIELD STRENGTH = 1.00 V/M   TIME = 0.0 SEC   NUMBER OF REGIONS = 6

RELATIVE DIELECTRIC CONSTANTS = 60.00   45.00   76.00   1.000   8.50   5.50   45.00

CONDUCTIVITIES (MHO/M) = 0.900   1.700   1.000   0.110   0.060   1.000

SURFACE BOUNDARIES (CM) = 5.270   5.470   5.520   5.800   5.900   6.000

Region		Radius	Theta	Phi	Absorbed Power Density
REGION 1	INTERIOR POINT:	RADIUS = 0.001 CM	THETA = 180.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.13498334 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 0.250 CM	THETA = 170.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.12557081 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 0.500 CM	THETA = 160.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.10592942 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 0.750 CM	THETA = 150.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.08574595 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 1.000 CM	THETA = 145.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.06549493 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 1.250 CM	THETA = 140.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.04931756 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 1.500 CM	THETA = 130.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.04162231 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 1.750 CM	THETA = 120.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.03448622 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 2.000 CM	THETA = 110.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.02676280 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 2.250 CM	THETA = 100.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.01936545 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 2.500 CM	THETA = 90.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.01374409 W/M***3
REGION 1	INTERIOR POINT:	RADIUS = 2.750 CM	THETA = 80.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.01027638 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 3.000 CM	THETA = 70.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00832354 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 5.470 CM	THETA = 60.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00649491 W/M***?
REGION 1	INTERIOR POINT:	RADIUS = 5.470 CM	THETA = 50.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00779011 W/M***?
REGION 2	INTERIOR POINT:	RADIUS = 5.520 CM	THETA = 40.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00439393 W/M***?
REGION 3	INTERIOR POINT:	RADIUS = 5.600 CM	THETA = 30.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00433350 W/M***?
REGION 4	INTERIOR POINT:	RADIUS = 5.800 CM	THETA = 20.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00222757 W/M***?
REGION 5	INTERIOR POINT:	RADIUS = 5.900 CM	THETA = 10.00 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00117700 W/M***3
REGION 6	INTERIOR POINT:	RADIUS = 6.000 CM	THETA = 0.0 DEG	PHI = 0.0 DEG	ABSORBED POWER DENSITY = 0.00693344 W/M***3

AVERAGE ABSORBED-POWER DENSITY = 1.6061BD-02 W/M***?

TOTAL ABSORBED POWER = 1.45324D-05 WATT

APPROXIMATE EXECUTION TIME = 0.05 CPU MINUTES

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APPENDIX B

SOURCE LISTING OF PROGRAM CSM

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C                           PROGRAM CSM          .                    CSM0001
C            ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC        CSM0002
C               SPHERICAL MODEL OF THE HUMAN OR ANIMAL HEAD           CSM0003
C                                                                     CSM0004
      IMPLICIT REAL*8 (A-H,O-Z)                                       CSM0005
      COMMON /COEFF/ANP,BNP,ALPNP,BETNP                               CSM0006
      COMMON FKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR,CSM0007
     1NC,NORG,NMIN                                                    CSM0008
      DIMENSION BDP(9),SBDP(9),EPSP(9),SIGP(9),P(101),DP(100)         CSM0009
      COMPLEX*16 FKP(10),CEX,ANP(1000),BNP(1000),ALPNP(1000),         CSM0010
     1BETNP(1000),BJNP(100),BHNP(100),Z                              CSM0011
      CALL ERRSET(208,0,-1,1)                                         CSM0012
      PIE=3.141592653589793D0                                         CSM0013
      RAD=180.D0/PIE                                                  CSM0014
      EPSO=8.85416D-12                                                CSM0015
      VEL=2.997924562D8                                               CSM0016
C ***      READ CONTROL PARAMETERS                                    CSM0017
    5 READ (5,10,END=110) FREQ,EO,TIME,STOPR,NORG,NOCP               CSM0018
   10 FORMAT(4E10.0,2I5)                                              CSM0019
C *** COMPUTE COMPLEX TIME VARIATION                                  CSM0020
      OMEGA=2.D6*PIE*FREQ                                             CSM0021
      ARG=-OMEGA*TIME                                                 CSM0022
      CEX=DCMPLX(DCOS(ARG),DSIN(ARG))                                 CSM0023
C ***      READ DIELECTRIC PROPERTY PARAMETERS                        CSM0024
      READ(5,20) (EPSP(I),SIGP(I),I=1,NORG)                          CSM0025
   20 FORMAT(8E10.0)                                                  CSM0026
C *** COMPUTE COMPLEX PROPAGATION CONSTANTS                           CSM0027
      FAC1=OMEGA/VEL                                                  CSM0028
      DO 30 I=1,NORG                                                  CSM0029
      FAC2=EPSP(I)/2.D0                                               CSM0030
      FAC3=DSQRT(1.D0+(1.D0/(EPSO*OMEGA)**2)*(SIGP(I)/EPSP(I))**2)   CSM0031
      REKP=FAC1*DSQRT(FAC2*(FAC3+1.D0))                              CSM0032
      FIMKP=FAC1*DSQRT(FAC2*(FAC3-1.D0))                             CSM0033
      FKP(I)=DCMPLX(REKP,FIMKP)                                      CSM0034
   30 CONTINUE                                                        CSM0035
      FKP(NORG+1)=DCMPLX(FAC1,0.D0)                                  CSM0036
C ***      READ RADII OF SURFACE BOUNDARIES                           CSM0037
      READ(5,20) (SBDP(I),I=1,NORG)                                  CSM0038
      DO 35 I=1,NORG                                                  CSM0039
      BDP(I)=SBDP(I)/1.D2                                             CSM0040
   35 CONTINUE                                                        CSM0041
C ***      PRINT OUT TITLE AND BASIC INPUT DATA                       CSM0042
      WRITE(6,40) FREQ,EO,TIME,NORG                                  CSM0043
   40 FORMAT('1ELECTROMAGNETIC ENERGY DEPOSITION IN A CONCENTRIC SPHERICSM0044
     1AL MODEL OF THE HUMAN OR ANIMAL HEAD'/'-FREQUENCY =',F9.2,' MHZ  CSM0045
     2   FIELD STRENGTH =',F7.2,' V/M        TIME =',F7.2,' SEC     NUMBECSM0046
     3R OF REGIONS =',I3)                                             CSM0047
      WRITE(6,41) (EPSP(I),I=1,NORG)                                 CSM0048
   41 FORMAT('0RELATIVE DIELECTRIC CONSTANTS =',9(F7.2,2X))          CSM0049
      WRITE(6,42) (SIGP(I),I=1,NORG)                                 CSM0050
   42 FORMAT('0CONDUCTIVITIES   (MHO/M)   =',9(F7.3,2X))             CSM0051
      WRITE(6,43) (SBDP(I),I=1,NORG)                                 CSM0052
   43 FORMAT('0SURFACE BOUNDARIES (CM) =',9(F7.3,2X))                CSM0053
C ***      COMPUTE SERIES EXPANSION COEFFICIENTS FOR ELECTRIC         CSM0054
C ***      FIELDS                                                     CSM0055
      CALL COEF(ANP,BNP,ALPNP,BETNP,NMIN)                            CSM0056
      WRITE(6,45)                                                     CSM0057
   45 FORMAT('0')                                                     CSM0058
      DO 70 I=1,NOCR                                                  CSM0059
C ***      READ DEFINING CHARACTERISTICS OF INTERIOR POINTS AT        CSM0060
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C ***        WHICH ABSORBED-POWER DENSITIES ARE TO BE COMPUTED            CSM0061
      READ(5,50)NREG,R,THETAD,PHID                                        CSM0062
   50 FORMAT(I5,3E10.3)                                                   CSM0063
      SAVR=R                                                              CSM0064
      P=R/1.D2                                                            CSM0065
      THETA=THETAD/RAD                                                    CSM0066
      PHI=PHID/RAD                                                        CSM0067
      Z=FKP(NREG)*R                                                       CSM0068
      CALL BJYH(BJNP,BHNP,Z,NC,STOPR)                                     CSM0069
      NC=NC+2                                                             CSM0070
      IF(NC.GT.NMIN)NC=NMIN                                               CSM0071
      CALL PL(THETA,NC,P,DP)                                              CSM0072
C ***        ABSORBED-POWER DENSITY AT GIVEN POINT INTERIOR TO P-TH REGIONCSM0073
      CALL EVEC(NREG,PD)                                                  CSM0074
      PD=.5D0*SIGP(NREG)*PD                                               CSM0075
C ***        PRINT OUT PARTICULARS OF INTERIOR POINT OF REGION P          CSM0076
      WRITE(6,60)NREG,SAVR,THETAD,PHID,PD                                 CSM0077
   60 FORMAT(' REGION',I2,' INTERIOR POINT: RADIUS =',F8.3,' CM  THETA =CSM0078
     1',F7.2,' DEG  PHI =',F7.2,' DEG  ABSORBED POWER DENSITY =',F12.8,'CSM0079
     2 W/M**3')                                                          CSM0080
   70 CONTINUE                                                            CSM0081
      NN=NORG*NMIN                                                        CSM0082
      FAC=2.D0*PIE/(FAC1*FAC1)                                            CSM0083
      QS=0.D0                                                             CSM0084
      QT=0.D0                                                             CSM0085
      DO 90 N=1,NMIN                                                      CSM0086
      FACN=2.D0*N+1.D0                                                    CSM0087
      QT=QT+FACN*DREAL(ALPNP(NN+N)+BETNP(NN+N))                           CSM0088
      QS=QS+FACN*(CDABS(ALPNP(NN+N))**2+CDABS(BETNP(NN+N))**2)            CSM0089
   90 CONTINUE                                                            CSM0090
      QA=FAC*(DABS(QT)-QS)                                                CSM0091
C ***        TOTAL ABSORBED POWER                                         CSM0092
      TOTPOW=2.65441D-3*EO**2*QA/2.D0                                     CSM0093
C ***        AVERAGE ABSORBED-POWER DENSITY                               CSM0094
      PAVG=TOTPOW/(4.D0*PIE*BDP(NORG)**3/3.D0)                            CSM0095
C ***        PRINT OUT AVERAGE ABSORBED-POWER DENSITY AND TOTAL ABSORBED  CSM0096
C ***        POWER                                                        CSM0097
      WRITE(6,100)PAVG,TOTPOW                                             CSM0098
  100 FORMAT('0',9X,'AVERAGE ABSORBED-POWER DENSITY =',1PD13.5,' W/M**3'CSM0099
     1/'0',9X,'TOTAL ABSORBED POWER =',D13.5,' WATT')                    CSM0100
      GO TO 5                                                            CSM0101
  110 STOP                                                               CSM0102
      END                                                                CSM0103
      SUBROUTINE COEF(ANP,BNP,ALPNP,BETNP,NMIN)                          CSM0104
C          GENERATES EXPANSION COEFFICIENTS                              CSM0105
      IMPLICIT REAL*8 (A-H,O-Z)                                          CSM0106
      COMMON FKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR,  CSM0107
     1NC,NORG                                                            CSM0108
      DIMENSION NTER(10),BDP(9),SIGP(9),P(101),DP(100)                   CSM0109
      COMPLEX*16 ANP(1000),BNP(1000),ALPNP(1000),BETNP(1000),BJHP1(1000)CSM0110
     1,BJHP2(1000),BJNP(100),BHNP(100),SJNP1(100),DELNP,SNT11,           CSM0111
     2SNT12,SNT21,SNT22,TNT11,TNT12,TNT21,TNT22,ETAP1,ETAP2,ZEP1,ZEP2,   CSM0112
     3SNP11,SNP12,SNP21,SNP22,TNP11,TNP12,TNP21,TNP22,DEL1,DEL2,FKP(10), CSM0113
     4CEX,RATIO,SHNP1(100),Z                                            CSM0114
C          COMPUTE EXPANSION COEFFICIENTS AN1,BN1,ANN,BNN,ALPN1,BETN1,   CSM0115
C          ALPNN,BETNN                                                   CSM0116
      N1=0                                                               CSM0117
      N2=0                                                               CSM0118
      DO 15 NR=1,NORG                                                    CSM0119
      Z=FKP(NR)*BDP(NR)                                                  CSM0120
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      CALL BJYH(BJNP,BHNP,Z,N,STOPR)                                    CSM0121
      DO 5 I=1,N                                                        CSM0122
      SJNP1(I)=BJNP(I)                                                  CSM0123
    5 SHNP1(I)=BHNP(I)                                                  CSM0124
      Z=FKP(NR+1)*BDP(NR)                                               CSM0125
      CALL BJYH(BJNP,BHNP,Z,NN,STOPR)                                   CSM0126
      NMIN=MINO(N,NN)                                                   CSM0127
      NTER(NR)=NMIN                                                     CSM0128
      N2=N2+NMIN                                                        CSM0129
      DO 10 I=1,NMIN                                                    CSM0130
      BJHP1(N1+I)=SJNP1(I)                                              CSM0131
      BJHP1(N2+I)=SHNP1(I)                                              CSM0132
      BJHP2(N1+I)=BJNP(I)                                               CSM0133
      BJHP2(N2+I)=BHNP(I)                                               CSM0134
   10 CONTINUE                                                          CSM0135
      N1=N1+2*NMIN                                                      CSM0136
      N2=N2+NMIN                                                        CSM0137
   15 CONTINUE                                                          CSM0138
      NMIN=MINN(NTER,NORG)                                              CSM0139
      NMIN=NMIN-2                                                       CSM0140
      DO 17 I=1,NMIN                                                    CSM0141
      ALPNP(I)=DCMPLX(0.D0,0.D0)                                        CSM0142
   17 BETNP(I)=DCMPLX(0.D0,0.D0)                                        CSM0143
      NSUM=NORG*NMIN                                                    CSM0144
      DO 30 I=1,NMIN                                                    CSM0145
      JJ=0                                                              CSM0146
      KK=0                                                              CSM0147
      II1=I+1                                                           CSM0148
      II2=2*I+1                                                         CSM0149
      SNT11=DCMPLX(1.D0,0.D0)                                           CSM0150
      SNT12=DCMPLX(0.D0,0.D0)                                           CSM0151
      SNT21=SNT12                                                       CSM0152
      SNT22=SNT11                                                       CSM0153
      TNT11=SNT11                                                       CSM0154
      TNT12=SNT12                                                       CSM0155
      TNT21=SNT12                                                       CSM0156
      TNT22=SNT11                                                       CSM0157
      DO 27 J=1,NORG                                                    CSM0158
      KK=KK+NTER(J)                                                     CSM0159
      ETAP1=(II1*BJHP1(JJ+I)-I*BJHP1(JJ+I+2))/II2                       CSM0160
      ETAP2=(II1*BJHP2(JJ+I)-I*BJHP2(JJ+I+2))/II2                       CSM0161
      ZEP1=(II1*BJHP1(KK+I)-I*BJHP1(KK+I+2))/II2                        CSM0162
      ZEP2=(II1*BJHP2(KK+I)-I*BJHP2(KK+I+2))/II2                        CSM0163
      DELNP=BJHP1(JJ+I+1)*ZEP1-BJHP1(KK+I+1)*ETAP1                      CSM0164
      RATIO=FKP(J+1)/FKP(J)                                             CSM0165
      SNP11=(ZEP1*BJHP2(JJ+I+1)-RATIO*BJHP1(KK+I+1)*ETAP2)/DELNP        CSM0166
      SNP12=(ZEP1*BJHP2(KK+I+1)-RATIO*BJHP1(KK+I+1)*ZEP2)/DELNP         CSM0167
      SNP21=(RATIO*BJHP1(JJ+I+1)*ETAP2-ETAP1*BJHP2(JJ+I+1))/DELNP       CSM0168
      SNP22=(RATIO*BJHP1(JJ+I+1)*ZEP2-ETAP1*BJHP2(KK+I+1))/DELNP        CSM0169
      Z=SNT11                                                           CSM0170
      SNT11=SNT11*SNP11+SNT12*SNP21                                     CSM0171
      SNT12=Z*SNP12+SNT12*SNP22                                         CSM0172
      Z=SNT21                                                           CSM0173
      SNT21=SNT21*SNP11+SNT22*SNP21                                     CSM0174
      SNT22=Z*SNP12+SNT22*SNP22                                         CSM0175
      TNP11=(RATIO*ZEP1*BJHP2(JJ+I+1)-BJHP1(KK+I+1)*ETAP2)/DELNP        CSM0176
      TNP12=(RATIO*ZEP1*BJHP2(KK+I+1)-BJHP1(KK+I+1)*ZEP2)/DELNP         CSM0177
      TNP21=(BJHP1(JJ+I+1)*ETAP2-RATIO*ETAP1*BJHP2(JJ+I+1))/DELNP       CSM0178
      TNP22=(BJHP1(JJ+I+1)*ZEP2-RATIO*ETAP1*BJHP2(KK+I+1))/DELNP        CSM0179
      Z=TNT11                                                           CSM0180
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      TNT11=TNT11*TNP11+TNT12*TNP21                               CSM0181
      TNT12=Z*TNP12+TNT12*TNP22                                   CSM0182
      Z=TNT21                                                     CSM0183
      TNT21=TNT21*TNP11+TNT22*TNP21                              CSM0184
      TNT22=Z     *TNP12+TNT22*TNP22                              CSM0185
      JJ=JJ+2*NTER(J)                                             CSM0186
      KK=KK+NTEF(J)                                               CSM0187
   27 CONTINUE                                                    CSM0188
      ANP(I)=SNT11-(SNT12*SNT21)/SNT22                            CSM0189
      BNP(I)=TNT11-(TNT12*TNT21)/TNT22                            CSM0190
      LL=NSUM+I                                                   CSM0191
      ANP(LL)=DCMPLX(1.D0,0.D0)                                   CSM0192
      BNP(LL)=DCMPLX(1.D0,0.D0)                                   CSM0193
      ALPNP(LL)=-SNT21/SNT22                                      CSM0194
      BETNP(LL)=-TNT21/TNT22                                      CSM0195
   30 CONTINUE                                                    CSM0196
      IF (NORG.EQ.1) RETURN                                       CSM0197
C         COMPUTE EXPANSION COEFFICIENTS AN2,...,AN(N-1);BN2,...,  CSM0198
C         BN(N-1);ALPN2,...,ALPN(N-1);BETN2,...,BETN(N-1)        CSM0199
      JJ=0                                                        CSM0200
      KK=0                                                        CSM0201
      MM1=0                                                       CSM0202
      MM2=NMIN                                                    CSM0203
      NRGM1=NORG-1                                                CSM0204
      DO 45 J=1,NRGM1                                             CSM0205
      KK=KK+NTER(J)                                               CSM0206
      DO 40 I=1,NMIN                                              CSM0207
      II1=I+1                                                     CSM0208
      II2=2*I+1                                                   CSM0209
      ETAP1=(II1*BJHP1(JJ+I)-I*BJHP1(JJ+I+2))/II2                CSM0210
      ETAP2=(II1*BJHP2(JJ+I)-I*BJHP2(JJ+I+2))/II2                CSM0211
      ZEP1=(II1*BJHP1(KK+I)-I*BJHP1(KK+I+2))/II2                 CSM0212
      ZEP2=(II1*BJHP2(KK+I)-I*BJHP2(KK+I+2))/II2                 CSM0213
      DELNP=BJHP1(JJ+I+1)*ZEP1-BJHP1(KK+I+1)*ETAP1              CSM0214
      RATIO=FKP(J+1)/FKP(J)                                       CSM0215
      SNP11=(ZEP1*BJHP2(JJ+I+1)-RATIO*BJHP1(KK+I+1)*ETAP2)/DELNP  CSM0216
      SNP12=(ZEP1*BJHP2(KK+I+1)-RATIO*BJHP1(KK+I+1)*ZEP2)/DELNP   CSM0217
      SNP21=(RATIO*BJHP1(JJ+I+1)*ETAP2-ETAP1*BJHP2(JJ+I+1))/DELNP CSM0218
      SNP22=(RATIO*BJHP1(JJ+I+1)*ZEP2-ETAP1*BJHP2(KK+I+1))/DFLNP  CSM0219
      DEL1=SNP11*SNP22-SNP12*SNP21                                CSM0220
      TNP11=(RATIO*ZEP1*BJHP2(JJ+I+1)-BJHP1(KK+I+1)*ETAP2)/DELNP  CSM0221
      TNP12=(RATIO*ZEP1*BJHP2(KK+I+1)-BJHP1(KK+I+1)*ZEP2)/DELNP   CSM0222
      TNP21=(BJHP1(JJ+I+1)*ETAP2-RATIO*ETAP1*BJHP2(JJ+I+1))/DELNP CSM0223
      TNP22=(BJHP1(JJ+I+1)*ZEP2-RATIO*ETAP1*BJHP2(KK+I+1))/DELNP  CSM0224
      DEL2=TNP11*TNP22-TNP12*TNP21                                CSM0225
      NN1=MM1+I                                                   CSM0226
      NN2=MM2+I                                                   CSM0227
      ANP(NN2)=(ANP(NN1)*SNP22-ALPNP(NN1)*SNP12)/DEL1           CSM0228
      BNP(NN2)=(BNP(NN1)*TNP22-BETNP(NN1)*TNP12)/DEL2           CSM0229
      ALPNP(NN2)=(-ANP(NN1)*SNP21+ALPNP(NN1)*SNP11)/DEL1        CSM0230
      BETNP(NN2)=(-BNP(NN1)*TNP21+BETNP(NN1)*TNP11)/DEL2        CSM0231
   40 CONTINUE                                                    CSM0232
      JJ=JJ+2*NTER(J)                                             CSM0233
      KK=KK+NTER(J)                                               CSM0234
      MM1=MM1+NMIN                                                CSM0235
      MM2=MM2+NMIN                                                CSM0236
   45 CONTINUE                                                    CSM0237
      RETURN                                                      CSM0238
      END                                                        CSM0239
      SUBROUTINE EVEC(NP,PD)                                      CSM0240
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C              COMPUTES THE RADIAL,COLATITUDE,AND AZIMUTHAL            CSM0241
C              COMPCNENTS OF ELECTRIC FIELD VECTOR E FOR              CSM0242
C              REGION P AND SCALAR PRODUCT E.E*                       CSM0243
       IMPLICIT REAL*8 (A-H,O-Z)                                      CSM0244
       COMMON /COEFF/ANP,BNP,ALPNP,BETNP                              CSM0245
       COMMON FKP,BJNP,BHNP,CEX,BDP,P,DP,SIGP,EO,TIME,R,THETA,PHI,STOPR, CSM0246
      1NC,NORG,NMIN                                                   CSM0247
       DIMENSION BDP(9),SIGP(9),P(101),DP(100)                        CSM0248
       COMPLEX*16 ERAD,ETHETA,EPHI,FKP(10),ANP(1000),BNP(1000),       CSM0249
      1ALPNP(1000),BETNP(1000),BJNP(100),BHNP(100),CEX,T,T1,PROD      CSM0250
       ERAD=DCMPLX(0.D0,0.D0)                                         CSM0251
       ETHETA=DCMPLX(0.D0,0.D0)                                       CSM0252
       EPHI=DCMPLX(0.D0,0.D0)                                         CSM0253
       NCK=0                                                          CSM0254
       NN=(NP-1)*NMIN                                                 CSM0255
       DO 40 N=1,NC                                                   CSM0256
       FAC1=2.D0*N+1.D0                                               CSM0257
       FAC2=N*(N+1.D0)                                                CSM0258
       RATIO=FAC1/FAC2                                                CSM0259
       T=FAC1*P(N)*(BNP(NN+N)*BJNP(N+1)+BETNP(NN+N)*BHNP(N+1))        CSM0260
       NCK=NCK+1                                                      CSM0261
       CALL TERM(NCK,T,1)                                             CSM0262
       ERAD=ERAD+T                                                    CSM0263
       T=ANP(NN+N)*BJNP(N+1)+ALPNP(NN+N)*BHNP(N+1)                    CSM0264
       CALL TERM(NCK,T,0)                                             CSM0265
       T1=BNP(NN+N)*((N+1.D0)*BJNP(N)-N*BJNP(N+2))/FAC1+BETNP(NN+N)*  CSM0266
      1((N+1.D0)*BHNP(N)-N*BHNP(N+2))/FAC1                            CSM0267
       CALL TERM(NCK,T1,1)                                            CSM0268
       IF((THETA.GE.1.D-6).AND.(THETA.LT.3.141592D0))GO TO 20         CSM0269
       IF(THETA.GE.3.141592D0)GO TO 10                                CSM0270
       ETHETA=ETHETA+FAC1/2.D0*T-RATIO*DP(N)*T1                       CSM0271
       EPHI=EPHI-RATIO*DP(N)*T+FAC1/2.D0*T1                           CSM0272
       GO TO 30                                                       CSM0273
   10  ETHETA=ETHETA+(-1.D0)**(N+1)*FAC1/2.D0*T-RATIO*DP(N)*T1        CSM0274
       EPHI=EPHI-RATIO*DP(N)*T+(-1.D0)**(N+1)*FAC1/2.D0*T1            CSM0275
       GO TO 30                                                       CSM0276
   20  ETHETA=ETHETA+RATIO*P(N)/DSIN(THETA)*T-RATIO*DP(N)*T1          CSM0277
       EPHI=EPHI-RATIO*DP(N)*T+RATIO*P(N)/DSIN(THETA)*T1              CSM0278
   30  IF(NCK.EQ.4)NCK=0                                              CSM0279
   40  CONTINUE                                                       CSM0280
       PROD=EO*CEX                                                    CSM0281
       ERAD=-PROD*DCOS(PHI)/(FKP(NP)*R)*ERAD                          CSM0282
       ETHETA=PROD*DCOS(PHI)*ETHETA                                   CSM0283
       EPHI=PROD*DSIN(PHI)*EPHI                                       CSM0284
       PD=DREAL(ERAD*DCONJG(ERAD))+DREAL(ETHETA*DCONJG(ETHETA))+DREAL(EPHCSM0285
      1I*DCONJG(EPHI))                                                CSM0286
       RETURN                                                         CSM0287
       END                                                           CSM0288
       SUBROUTINE TERM(NCK,T,KEY)                                     CSM0289
C              COMPUTES I**NCK*(N-TH TERM IN SERIES)                  CSM0290
       IMPLICIT REAL*8 (A-H,O-Z)                                      CSM0291
       COMPLEX*16 T                                                   CSM0292
       IF(KEY.EQ.1)GO TO 20                                           CSM0293
       GO TO (5,10,15,45),NCK                                         CSM0294
   20  GO TO (10,15,45,5 ),NCK                                        CSM0295
    5  T=DCMPLX(-DIMAG(T),DREAL(T))                                   CSM0296
       GO TO 45                                                       CSM0297
   10  T=-T                                                           CSM0298
       GO TO 45                                                       CSM0299
   15  T=DCMPLX(DIMAG(T),-DREAL(T))                                   CSM0300
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   45 RETURN                                                            CSM0301
      END                                                               CSM0302
      SUBROUTINE BJYH(BJNP,BHNP,Z,NN,STOPR)                             CSM0303
C           GENERATES SPHERICAL BESSEL FUNCTIONS JN(KR) AND YN(KR)      CSM0304
C           AND SPHERICAL HANKEL FUNCTIONS OF THE FIRST KIND HN(KR)     CSM0305
      IMPLICIT REAL*8 (A-H,O-Z)                                         CSM0306
      COMPLEX*16 BJNP(100),BYNP(100),BHNP(100),QP,Z                     CSM0307
      BYNP(1)=-CDCOS(Z)/Z                                               CSM0308
      BYNP(2)=(BYNP(1)-CDSIN(Z))/Z                                      CSM0309
      DO 5 M=3,100                                                      CSM0310
      BYNP(M)=(2*M-3)/Z*BYNP(M-1)-BYNP(M-2)                             CSM0311
      IF(CDABS(BYNP(M)).GE.STOPR)GO TO 10                               CSM0312
    5 CONTINUE                                                          CSM0313
   10 IF(M.GT.3)GO TO 25                                                CSM0314
C ***         PRINT OUT ERROR MESSAGE                                   CSM0315
      WRITE(6,20)Z                                                      CSM0316
   20 FORMAT('0**** PROCESS CAN NOT PROCEED SINCE NM2 = 0 FOR Z =',1P2D1CSM0317
     15.7)                                                              CSM0318
      STOP                                                              CSM0319
   25 BJNP(M)=DCMPLX(0.D0,0.D0)                                         CSM0320
      BJNP(M-1)=-1.D0/(Z*Z*BYNP(M))                                     CSM0321
      NM2=M-2                                                           CSM0322
      DO 30 I=1,NM2                                                     CSM0323
      L=M-I                                                             CSM0324
      BJNP(L-1)=(2*L-1)/Z*BJNP(L)-BJNP(L+1)                             CSM0325
   30 CONTINUE                                                          CSM0326
      QP=CDSIN(Z)/(Z*BJNP(1))                                          CSM0327
      NM1=M-1                                                           CSM0328
      DO 35 N=1,NM1                                                     CSM0329
      NN=N                                                              CSM0330
      BJNP(N)=QP*BJNP(N)                                                CSM0331
      IF(CDABS(BJNP(N)).LT.1.D-25)GO TO 40                              CSM0332
   35 CONTINUE                                                          CSM0333
   40 DO 45 I=1,NN                                                      CSM0334
      REJN=DREAL(BJNP(I))                                               CSM0335
      FIMJN=DIMAG(BJNP(I))                                              CSM0336
      REYN=DREAL(BYNP(I))                                               CSM0337
      FIMYN=DIMAG(BYNP(I))                                              CSM0338
      REHN=REJN-FIMYN                                                   CSM0339
      FIMHN=REYN+FIMJN                                                  CSM0340
      BHNP(I)=DCMPLX(REHN,FIMHN)                                        CSM0341
   45 CONTINUE                                                          CSM0342
      RETURN                                                            CSM0343
      END                                                               CSM0344
      SUBROUTINE PL(THETA,N,P,DP)                                       CSM0345
C           GENERATES ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST        CSM0346
C           KIND, ORDER 1 AND DEGREE N, AND THEIR FIRST DERIVATIVES     CSM0347
      IMPLICIT REAL*8 (A-H,O-Z)                                         CSM0348
      DIMENSION P(101),DP(100)                                          CSM0349
      SNJ=DSIN(THETA)                                                   CSM0350
      CNJ=DCOS(THETA)                                                   CSM0351
      P(1)=SNJ                                                          CSM0352
      P(2)=3.D0*SNJ*CNJ                                                 CSM0353
      DP(1)=CNJ                                                         CSM0354
      DO 10 M=2,N                                                       CSM0355
      A=M                                                               CSM0356
      MP1=M+1                                                           CSM0357
      P(MP1)=(2.D0*A+1.D0)/A*CNJ*P(M)-(A+1.D0)/A*P(M-1)                 CSM0358
      IF((THETA.GE.1.D-6).AND.(THETA.LT.3.141592D0))GO TO 5            CSM0359
      DP(M)=M*MP1/2                                                     CSM0360
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      IF(THETA.GE.3.141592D0)DP(M)=(-1.D0)**M*DP(M)            CSM0361
      GO TO 10                                                 CSM0362
    5 DP(M)=(A*CNJ*P(M)-(A+1.D0)*P(M-1))/SNJ                   CSM0363
   10 CONTINUE                                                 CSM0364
      RETURN                                                   CSM0365
      END                                                      CSM0366
      FUNCTION MINN(NRAY,N)                                    CSM0367
C         DETERMINES MINIMUM POSITIVE INTEGER VALUE            CSM0368
      DIMENSION NRAY(10)                                       CSM0369
      IF(N.EQ.1)GO TO 20                                       CSM0370
      NMIN=NRAY(1)                                             CSM0371
      DO 10 I=2,N                                              CSM0372
      NTEMP=NRAY(I)                                            CSM0373
      IF(NTEMP.LT.NMIN)NMIN=NTEMP                              CSM0374
   10 CONTINUE                                                 CSM0375
      MINN=NMIN                                                CSM0376
      GO TO 30                                                 CSM0377
   20 MINN=NRAY(1)                                             CSM0378
   30 RETURN                                                   CSM0379
      END                                                      CSM0380
```